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Features

Death and statistics

by John Webb

Financial planning for the future

With improving health, most of the population of Britain can expect to live many years after they take retirement from formal employment in their sixties. After that, they will have to live on a social security pension, or on a pension from their own investments and private pension funds. Planning for retirement is not usually top of somebody's list when they are starting their first job straight out of school or university. But it is an important issue that cannot be ignored. In the past, a salary package in a big company would include compulsory contributions to a pension fund, over which the new employee did not have much control. But today fewer people expect to be in one big company for all their working life. More typically, people are likely to change jobs several times in their careers. They may then need to transfer pension fund contributions from one company to another, and may find that this is not as simple as it seems. Others set up their own businesses, and must make independent provision for their retirement. In such circumstances careful financial planning for the future has to be done right from the start.

In the Issue 11 of *Plus* the article *Have we caught your interest?* showed that an understanding of the mathematics of compound interest is essential to making the right short−term investment decisions. This article goes further: it examines the role of actuarial mathematics in long−term planning for the future.

What is an annuity?

When you invest an amount of (for example) £1000 at 4% interest per annum, you can expect to be paid an *annuity* of £40 every year from then on. If you die, the annuity carries on being paid to your heirs, or the original £1000 investment is refunded. That is the sort of annuity offered by a bank.

A life assurance company offers a different sort of annuity. In exchange for your investment of £1000, the company will agree to pay you a fixed amount every year until your death, upon which the payments cease, and the capital is not refunded. This is a *life annuity*. The amount paid will be significantly more than the £40 you would get from the bank, and the older you are when you purchase the annuity, the more the company will offer to pay you. A life annuity is an attractive option for somebody who retires at the age of 65, for example, and wants a guaranteed income for the rest of his or her life.

How does a life assurance company calculate the annuity payment? Two key factors have to be taken into consideration. The first is compound interest: the amount of money that can be earned from the original payment. This type of calculation was discussed in "*Have we caught your interest?*"?

The second issue is how long the purchaser of the annuity is expected to survive. This is a matter of probability and statistics. Actuarial science begins at this confluence of two branches of mathematics: the combination of compound interest with observed mortality statistics. The actuarial profession uses mathematics to make financial sense of the future.

Annuities in history

Financial agreements that depend on the duration of somebody's life date back hundreds of years. A typical example would be a lease on a property, which might be agreed to last until the death of the tenant.

The idea of selling life annuities seems to have originated in the state rather than the private sector. While financing their day−to−day running from taxes, governments have always needed to raise loans for large capital expenses or to finance military campaigns. In the 17th century, governments hit on the idea of selling life annuities in order to raise capital.

In the early days of selling annuities, the price of an annuity was often calculated by rule of thumb. For example, the cost of an annuity might be set at 14 times the annual payment, regardless of the age of the purchaser of the annuity. This would be an attractive for a young person with capital to spare, but rather less so for a pensioner aged 65.

The mathematical prime minister

The first person to make actuarial calculations combining compound interest and mortality rates was Johan de Witt, Prime Minister of the Netherlands from 1653 to 1672.

Johann de Witt had a distinguished mathematical background. He was born in the period just before Isaac Newton (1642−1727) and Gottfried Leibniz (1646−1716) revolutionized mathematics with the introduction of differential and integral calculus. De Witt was the author of *"The elements of curves"*, the first treatise on conic sections to use Descartes' recently−developed coordinate geometry. (It was de Witt, by the way, who coined the term *directrix*.)

The theory of probability was also at the forefront of mathematical development in the first half of the 17th century. When de Witt moved into government he was thus well prepared to apply his understanding of probability to financial mathematics, and wrote a book entitled *"The worth of life annuities compared to redemption bonds"*.

De Witt was one of the great statesmen of his day, at a time when the Netherlands was developing as a major trading nation and posing a serious challenge to the naval power of England. De Witt forged important treaties with England and Sweden and under his leadership the Netherlands enjoyed great commercial prosperity. But he fell out of favour when France invaded the Netherlands in 1672 and was murdered by an angry mob.

To calculate the cost of an annuity de Witt devised a formula based on a hypothetical group of 768 lives. He assumed that they would die according to the following rule, which today would be termed a *mortality table*:

6 die every six months for the first 50 years
4 die every six months for the next 10 years
3 die every six months for the next 10 years
2 die every six months for the next 7 years.

Using 4% compound interest tables, De Witt then calculated the present value of the series of payments to be made to a purchaser of an annuity, for each of the possible ages of death. For example, for the age at death of 62 years and six months, the present value of the 125 six−monthly payments of an amount A would be

$$A(1.02^{-1} + 1.02^{-2} + \ldots + 1.02^{-125}) = 45.79A$$

(summing the geometric series).

The present value of an amount A to be paid in n years time is the amount that would have to be invested now at an interest rate of r to reach the amount A in n years. So if the present value is P, then

$$P(1+r)^n = A,$$

so that $P = (1+r)^{-n}$.

For using the concept of present value in compound interest calculations, see "*Have we caught your interest?*".

The sum of all 768 present values, divided by 768, gave the average value of one annuity, bought at birth. This was then set as the price of an annuity.

De Witt's mortality figures are far too tidy to have been derived by observation alone, although they may have been influenced by actual records of births and deaths.

The mathematical mayor

Another Dutchman who moved from a mathematical career into politics was a contemporary of de Witt, Johan Hudde (1628−1704). His mathematical work covered Cartesian geometry and the theory of equations. He built microscopes and telescopes, and was an expert in canals engineering. Dutch military strategy included breaching dykes to hamper invading armies, and it was Hudde who directed the flooding of areas of

Holland as a defensive strategy against the French invasion of 1672.

Johann Hudde was burgomaster (mayor) of Amsterdam for 30 years, from 1672 to 1702. To raise funds, the Amsterdam city government took to selling annuities. Using statistics of a group of 1495 people who had purchased annuities in the period 1586−1590 and were now all dead, Hudde applied his mathematical skills (in particular, his knowledge of the theory of probability) to calculate annuity prices. It is important to recognize that Hudde's mortality table was derived from a genuine statistical survey, unlike de Witt's, which was a theoretical construct. It is also significant that Hudde's mortality table was based on the death rates of people who had bought annuities, rather than the death rates of the general population.

Again using an interest rate of 4%, Hudde calculated purchases prices of annuities. He recommended, for example, a purchase price of 17.2 times the annual payment for an annuity to a six−year−old.

It is an interesting little exercise at this stage to use this information to work backwards to find out how long a six−year−old child could be expected to live in 17th century Europe.

Suppose the annual payment is A, and the annual interest rate is r. Then the present value of n annual payments is

$$A((1+r)^{-1} + (1+r)^{-2} + \ldots + (1+r)^{-n})$$

$$= \mathrm{A}\frac{}{r(1-(1+r)^{-n})}$$

(summing the geometric series).

Putting $r = 1.04$ and equating the above expression to $17.2 \times\ A$, the value of n is found to be 29.7. In other words, Huddeâ€™s mortality table predicted that a child aged 6 would on average survive to the age of 35.7 years.

The work of Edmond Halley

Across the Channel, a very influential work on the calculation of life annuities was published in the Philosophical Transactions of the Royal Society in 1693. The author was the astronomer Edmond Halley, who is far better known today for having a famous comet named after him. Halley analysed records of deaths from the city of Breslau, in Germany. Starting with a population of 1000 at age 1, Halley worked out the numbers surviving at each age up to 84 (the figures for the first 8 years are given below).

Age:	1	2	3	4	5	6	7	8
Number:	1000	855	798	760	732	710	692	680

These figures reflect the high rate of infant mortality in the 17th Century. The figures show that nearly a third of children age 1 fail to reach their eighth birthday.

Suppose one wants to calculate the value of a life annuity of A per annum to someone aged 5 years. The present value of the first payment is

$$A \times\ 1.04^{-1} = 0.9615A.$$

Since the probability of a child age five surviving to age six is $\frac{710}{732}$, the net present value of the first payment is

$$A \times \ 1.04^{-1} \times \ \frac{710}{732} = 0.9326A.$$

The net present value of the second payment is calculated similarly, multiplying the present value of the second payment by the probability of survival from age 5 to age 7:

$$A \times \ 1.04^{-2} \times \ \frac{692}{732} = 0.8740A,$$

and the net present value of the third payment is similarly

$$A \times \ 1.04^{-3} \times \ \frac{680}{732} = 0.8258A.$$

This calculation is continued for every year until the end of the mortality table, and then all the net present values are added together. The sum gives the price of the annuity.

The basic life annuity has many variations, and Halley considered the problem of selling annuities dependent on two lives. This could have pitfalls. In the 1690s the Mercer's Company in London offered an annuity whereby a husband would pay a premium of £100. After his death, his widow would receive £30 per annum for the rest of her life. The calculation was based on an interest rate of 6%, and assumed that the couple were roughly the same age. The scheme ran into difficulties when the interest rate fell. Moreover, the scheme had admitted a number of elderly men with much younger wives, which common sense should have suggested would lead to a major long−term drain on the scheme's reserves. Indeed, the scheme eventually had to be closed down and rescued by the government.

The amount of labour involved in calculating the price of an annuity is considerable. The problem is that the mortality table numbers do not conform to a simple formula. If they did, the calculations could be shortened by judicious use of algebra. A political refugee in London proposed a solution.

The Huguenot simplification

Abraham de Moivre (1667−1754) was a French Huguenot who took refuge in England from religious persecution in 1685. Apart from his work in probability and actuarial science, he is remembered today for his work in complex numbers, and in particular for the formula

$$(\cos\ x + i \sin\ x)^n = \cos\ nx + i \sin\ nx$$

(where i is the square root of minus one).

Because of prejudice against foreigners, De Moivre was never able to obtain a university appointment in England, although his mathematical prowess brought him election as a Fellow of the Royal Society in 1697. He had to earn a living by tutoring mathematics, and died in poverty. In his work on Halley's mortality table, De Moivre introduced the assumption that the number of people living decreased in arithmetical progression, i.e. that with an initial population of N, and d people dying every year, the number of survivors after k years would be $N - kd$. It turned out that de Moivre's results were sufficiently close to those of Halley to be of practical use.

There is an anecdote about De Moivre that, in view of his application of arithmetic progressions to the calculation of mortality tables, has a particular poignancy. He was 87 years old, and noticed that every night he slept a quarter of an hour longer than the previous night. This suggested to him that his death would occur when he reached 24 hours of sleep, and accordingly predicted the date of his death. His prediction was accurate.

The arithmetic−geometric series

Although de Moivre's simplification is no longer used in mortality calculations nowadays, it is nevertheless worth looking at as an interesting mathematical exercise.

Using de Moivreâ€™s formula, the probability of an individual in an original population of N surviving for n years in $(N-nd)/N$, for $n < N/d$ (we assume that N/d is an integer). The net present value of the nth annual payment of an amount A to that person is

$$A \times \ (1+r)^{-n} \times \ \frac{N-nd}{N}.$$

These terms must be summed for $n = 1, 2, 3, \ldots, N/d$.

The algebraic problem that now arises is to find a formula for the sum of a series of the form

$$1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}.$$

This series, part arithmetic and part geometric, is no longer encountered in school mathematics (although it was part of the staple diet of algebra textbooks a hundred years ago. A formula for the sum of *n* terms of this series can be obtained by the same method used for summing the ordinary geometric series:

- Let $S = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$.
- Multiply both sides by x:
  $xS = x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n$.
- Subtract the second series from the first and sum the geometric series:
  $$x - xS = 1 + x + x^2 + \ldots + x^{n-1} - nx^n = \frac{1-x^n}{1-x} - nx^n.$$

  Thus

  $$S = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}.$$

  Another way of obtaining this result is to differentiate both sides of the formula for the sum of a geometric progression:

  $$1 + x + x^2 + x^3 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}.$$

  The details are left as an exercise.

  To end this mathematical diversion, just two comments. The first is that the calculation has been carried out under the assumption that x is not equal to 1 (which is assured, because in our calculations $x = 1 + r$). The second comment is that, like the geometric series, the arithmetic−geometric series converges (can be summed to infinity) if and only if $-1 < x < 1$.

The way ahead was now clear for actuarial science, which made major progress in the first half of the 18th century. Life policies and pension schemes were developed, and the value of a good mortality table was recognized. It was seen that actuaries needed a strong blend of mathematics, statistics and financial acumen to achieve their aim: to make financial sense of the future.

Reference

Life, death and money: actuaries and the creation of financial security. Derek Renn (editor). Blackwell 1998.

About the author

John H. Webb was born in Cape Town and studied mathematics at the University of Cape Town. He won a scholarship to Cambridge where he obtained a Ph D. Back at the University of Cape Town, his career as a research mathematician was eventually overtaken by interests in mathematics education, with particular emphasis on popularising mathematics and identifying and stimulating promising students.

He edits Mathematical Digest, a quarterly magazine for high schools, runs a maths competition for schools in the Cape Town area, and directs a nationwide Mathematical Talent Search, a problem−solving programme by correspondence which selects and trains South African teams for the International Mathematical Olympiad.

He is at present on sabbatical leave, and is spending six months working with the Millennium Mathematics Project in Cambridge. His visit is supported by the Institute of Actuaries.

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.