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Features

Modelling, step by step

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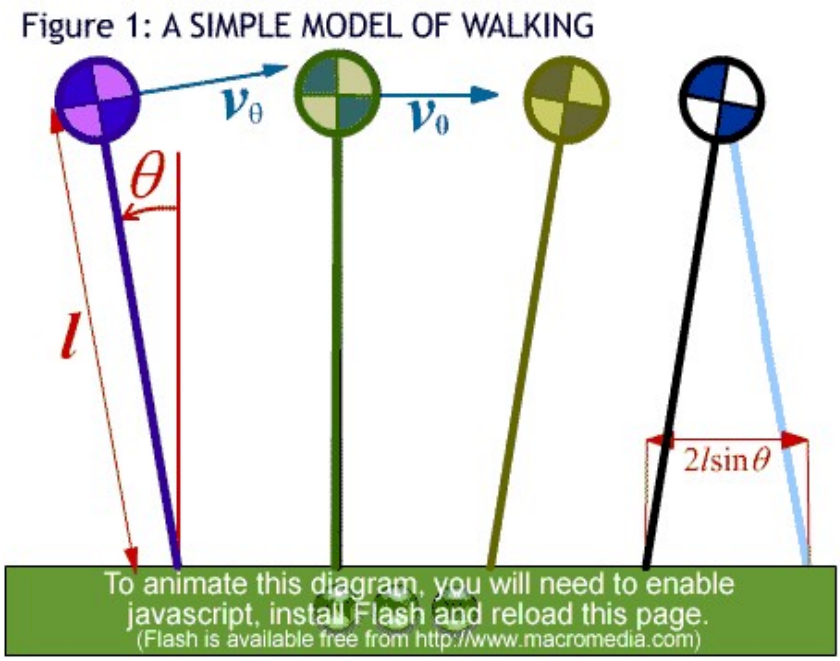
I am walking to the bus stop when I see a bus approaching, so I walk faster and faster still. At a certain speed (about 2 metres per second, if you take the trouble to measure it) I will find that I want to start running. Perhaps, however, I decide that on this occasion I prefer not to run, but to walk faster still. As my speed approaches 3 metres per second I will find that I simply cannot walk faster. Those speeds of 2 and 3 metres per second are not peculiar to me, but apply to normal−sized human adults in general. They apply as much to students as to elderly professors. Why is walking limited like this? Can mathematics help us to understand?

The obvious approach is to devise a mathematical model of the human body, and to see what happens when it moves at different speeds. That may seem a daunting task, because the human body is very complicated. In one leg alone I have 29 bones and 37 muscles. Can we nevertheless do some simple mathematics that will help us to understand how we move? I believe that we can.



First, we need to remind ourselves what walking is like. One foot is set down just before the other is lifted. While a foot is on the ground, the knee of the same leg remains more or less straight. As we walk, our heads bob up and down. The centre of mass of the body is between, and a little above, the hip joints.

Figure 1 shows a model of walking that moves like this. The legs are rods that remain straight while the foot is on the ground. The centre of mass of the body is at the hip joints. Each foot is set down as the other is lifted, and the body moves forward in a series of arcs of circles.

A body moving in a circle has an acceleration towards the centre of the circle. If its speed is v and the radius of the circle is r, the acceleration is v^2/r. Let the walkerâ€™s body have velocity v_0, at the stage of the stride when the leg is vertical. Then at this stage the body has a downward acceleration v_0^2/l, where l is the length of the legs. The walkerâ€™s feet are not glued to the floor, so the body cannot be pulled down; it can only fall under gravity. Consequently, the downward acceleration cannot be greater than the gravitational acceleration g:

$$g \geq v_0^2/l$$

$$v_0 \leq \sqrt{gl} \qquad (1)$$

My legs are about 0.9m long. The gravitational acceleration is about $10\mathrm{m/s}^2$. Thus equation (1) tells me that I cannot walk faster than $\sqrt{10 \times 0.9} = 3$ m/s, which is one of the things we set out to explain. The equation predicts only slightly different speed limits for normal adults of different statures, but lower speeds for small children. For example my grandson Sam has legs about 0.4m long, so his walking speed limit is about 2m/s. If I walk fast, Sam has to run to keep up.

The equation also predicts that the walking speed limit will be lower in reduced gravity, for example on the moon or on Mars. Experiments in aircraft have confirmed that the speed of changing from walking to running is reduced, when low gravity is simulated by flying a parabolic trajectory.

If you are knowledgeable about athletics, you may want to object at this point. Surely athletes in walking races go faster than 3m/s? They do; the world records for the 10km walk require mean speeds of 4.4m/s for men and 4.0m/s for women. The explanation is that athletes wiggle their hips in the racing walk, in such a way as to reduce the vertical movements of the bodyâ€™s centre of mass. Their centres of mass rise and fall less than their hip joints, so one of the assumptions of our model is violated.

We seem to have explained why we cannot go faster than 3m/s by normal walking, but we would like to explain also why we prefer to start running well before we reach that speed. One plausible hypothesis is that we can save energy by breaking into a run. An experiment confirms this. Physiologists have measured the rates at which people use oxygen, when walking or running at different speeds. The results have shown that at speeds up to 2m/s, walking uses less energy than running; but above 2m/s, running is the more economical gait. Can mathematics help us to understand why this should be?

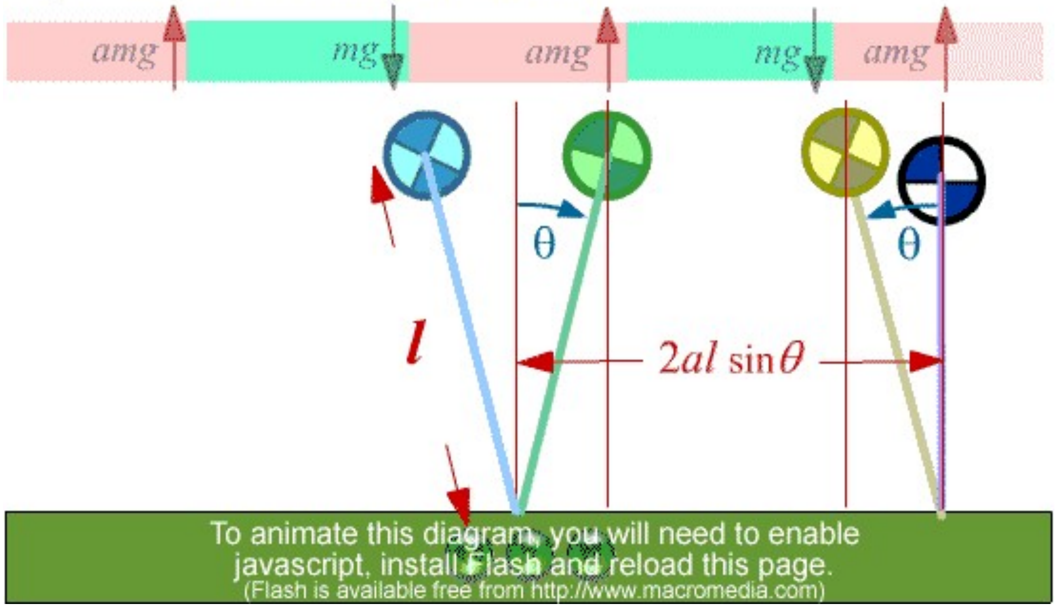
While only one foot is on the ground, and the body is moving along a circular arc, no work is required. The body slows down a little as it rises, and speeds up again as it falls. Kinetic energy is converted to gravitational potential energy and back again, as in a swinging pendulum. Work is required only at the instant when one foot hits the ground and the other leaves it, at which stage both legs make angles θ with the vertical. Immediately before this instant, the centre of mass is travelling with velocity v_θ at an angle $-\theta$ to the horizontal (v_θ is a little greater than v_0, because some potential energy has been converted to kinetic energy). Immediately after this instant, the centre of mass is travelling with velocity v_θ at an angle $+\theta$ to the horizontal. The vertical component of its velocity changes from $-v_\theta \sin(\theta)$ to $+v_\theta \sin(\theta)$. The mass of the body is m, so it loses and regains kinetic energy amounting to $\frac{1}{2} m v_\theta^2 \sin^2(\theta)$. The walkerâ€™s muscles must do this amount of work, to replace the lost kinetic energy, in every step. In a step, the walker travels a distance $2l \sin\theta$. Thus the work W_{walk} required to walk unit distance is

$$W_{walk} = (m v_\theta^2 sin\theta)/4l \qquad (2)$$

Now we will estimate the work needed for running. When we run, there are times when both feet are simultaneously off the ground. We bend the knee a little, as the body passes over the supporting foot. While a foot is on the ground, the force on it remains more or less in line with the leg.

In walking, kinetic energy is converted to gravitational potential energy and back again, as in a pendulum. In running, however, kinetic and potential energy fluctuate in phase with each other. The body is highest and moving fastest when both feet are off the ground. Consequently, the pendulum principle cannot operate.

Figure 2: A SIMPLE MODEL OF RUNNING



The distance travelled between one footfall and the next is now greater than $2l\sin\theta$; let it be $2al\sin\theta$. The mean force on the ground over a complete stride must equal the walkerâ€™s weight, mg. There is a foot on the ground for only a fraction $1/a$ of the time, so while a foot is on the ground it must exert a mean vertical force amg. Suppose that the vertical component of the force on the ground is constant and equal to amg. Then to keep the resultant force in line with the leg, there must be a horizontal component of force equal to $amg\tan\theta$ when the foot is first set down, falling to zero as the leg becomes vertical. The body moves forward a distance $l\sin\theta$ against a mean force of about $0.5amg\tan\theta$, so it loses kinetic energy $0.5lamg\tan\theta\sin\theta$. The runnerâ€™s muscles must do work to replace this amount of energy, every time he or she travels $2al\sin\theta$. Thus the work done per unit distance is

$$W_{run} = (mg\tan\theta)/4 \qquad (3)$$

Equation (2) tells us that the work required to walk unit distance increases in proportion to the square of the speed, and equation (3) tells us that the work needed to run unit distance is independent of speed. Therefore, there must be a speed above which running is more economical than walking.

Now let us use the equations to estimate the speed at which running becomes the more economical gait. At this critical speed, $v_{\theta\mathrm{crit}}$, W_{walk} equals W_{run}, so equations (2) and (3) tell us

$$\begin{aligned} (mv_{\theta\mathrm{crit}}^2\sin\theta)/4l &= (mg\tan\theta)/4 \\ v_{\theta\mathrm{crit}} &= \sqrt{\left(\frac{gl}{\cos\theta}\right)}. \qquad (4) \end{aligned}$$

The angle θ is typically about 25^o for human walking, so $\cos\theta$ is about 0.9. Using our estimate of 0.9m for leg−length, this leads to the estimate

$$v_{\theta\mathrm{crit}} = \sqrt{10}. \qquad (5)$$

Now $v_{\theta\mathrm{crit}}$ is the velocity of the walkerâ€™s centre of gravity when both legs make maximum angle with the vertical. This velocity fluctuates a little in the course of a walking stride because kinetic energy is converted to gravitational potential energy and back again, as in a pendulum. Consequently, v_θ is a little greater than v_0. (Can you work out how much greater?) When account is taken of this, equation (4) predicts that the speed at which it becomes more economical to run is a little less than the maximum possible walking speed that is given by equation (1).

However, I must admit at this stage that our calculations have been too simple to calculate this speed accurately. Their principal shortcomings are

- In deriving equation (3), we took account of the work required to replace lost kinetic energy, but ignored the (smaller) quantity needed to replace lost potential energy.
- We assumed that the vertical component of the force on the ground, in running, is constant throughout the time that the foot is on the ground. In fact, it rises to a peak and falls again.
- We ignored the elasticity of tendons and ligaments. The Achilles tendon, and the ligaments of the arch of the foot, function as springs; they store up some of the kinetic and potential energy that a runner loses in the first half of a step, and return it in an elastic recoil. Thus to some extent a runner bounces along like a rubber ball. Experiments on the elasticity of tendons and of amputated feet indicate that about half the work that would otherwise be needed for running is saved in this way.
- We assumed that the force on a foot acts at a fixed point. Force records made with instrumented panels set into the floor show that the centre of pressure starts at the heel and moves forward to the toes, in the course of a step.
- We calculated mechanical work, but it would have been better to have calculated fuel consumption, the quantity of food energy that is used. This would not have been simple, because the efficiency with which muscles convert food energy into mechanical work depends on the rate at which they are contracting.

Not surprisingly, mathematical models that take account of all these things are more accurate than the simple one that we have used, in explaining the speed at which we change from walking to running.

That is not an argument against using very simple models. The important point that I want to emphasise is that we have gained useful insight, for very little mathematical effort. It is generally best, in studying human and animal movement, to start with an exceedingly simple model. You can then make rough estimates of the errors that have been introduced by your simplifying assumptions, and elaborate the model to eliminate the more serious errors. Keeping models simple will not only save you work, but it will also help you to see more clearly the principles that the models illustrate.

As well as the one that we have discussed, human movement raises many problems that can be tackled by mathematical modelling. A very simple model (a stick man with a single leg muscle) has been used to explain why high jumpers run up much more slowly than long jumpers. Another explains why athletes performing a standing jump can jump higher if they make a countermovement, bending their knees immediately before extending them. Others have been used to calculate how far a walker, who wants to use as little energy as possible, should be prepared to deviate from a straight path to avoid a hill or a patch of marshy ground. There is a great deal of scope for simple mathematical modelling, in this and other areas of biology.

Finally, did you manage to calculate how much a walkerâ€™s speed fluctuates in the course of a step? The gravitational potential energy lost as the leg moves from its vertical position in mid−step, to its angle θ at the end of the step, is $mgl(1-\cos\theta)$. The kinetic energy that it gains is $\frac{1}{2}m(v_\theta^2-v_0^2)$. The kinetic energy gained equals the potential energy lost, so

$$\begin{aligned} \frac{1}{2}m(v_\theta^2 - v_0^2) &= mgl(1-\cos\theta), \\ v_\theta^2 &= v_0^2[1+(2gl/v_0^2)(1-\cos\theta)]. \end{aligned}$$

We have seen that $cos\theta$ is about 0.9. Thus v_θ is about $1.1v_0$ at the maximum possible walking speed, when $v_0 = \sqrt{(gl)}$.

Using the estimate of $\sqrt{10}$ found above for $v_{\theta\mathrm{crit}}$, we estimate that the speed above which running becomes more comfortable and energy−efficient than walking is $\sqrt{10}/1.1$ = 2.9m/s. In fact, the experimental value is 2.0m/s, which is also the answer given by models taking into account the bullet points listed above.

About the author

Professor R McNeill Alexander, FRS, is Emeritus Professor of Zoology in Leeds. Until recently he was Secretary of the Zoological Society of London and President of the Society for Experimental Biology.

His research on the mechanics of human and animal movement has won him many honours including Fellowship of the Royal Society, Honorary Membership of the American Society of Zoologists and medals from the Zoological Society, the Linnean Society and the International Society for Biomechanics.

He is the author of thirteen books including *Animal Mechanics*, *Dynamics of Dinosaurs*, *Animals* and *The Human Machine.*

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.