- about *Plus*
- support *Plus*
- subscribe to *Plus*
- terms of use

search *plus* with google

- home
- latest issue
- explore the archive
- careers library
- news

November 2003
Features

The prime number lottery

by Marcus du Sautoy

David Hilbert

On a hot and sultry afternoon in August 1900, David Hilbert rose to address the first International Congress of Mathematicians of the new century in Paris. It was a daunting task for the 38 year−old mathematician from Göttingen in Germany. You could hear the nerves in his voice as he began to talk. Gathered in the Sorbonne were some of the great names of mathematics. Hilbert had been worrying for months about what he should talk on. Surely the first Congress of the new century deserved something rather more exciting than just telling mathematicians about old theorems.

So Hilbert decided to deliver a very daring lecture. He would talk about what we didn't know rather what we had already proved. He challenged the mathematicians of the new century with 23 unsolved problems. He believed that "Problems are the life blood of mathematics". Without problems to spur the mathematician on in his or her journey of discovery, mathematics would stagnate. These 23 problems set the course for the mathematical explorers of the twentieth century. They stood there like a range of mountains for the mathematician to conquer.

As the century came to a close, all the problems had essentially been solved. All except one: the Riemann Hypothesis. The Everest of all Hilbert's problems. It was in fact Hilbert's favourite problem. When asked what would be the first thing he would do if he were brought to life again after 500 years he said "I would ask whether the Riemann Hypothesis has been proved". All indications are that the answer might still be "no" for it appears to be one of the hardest problems on the mathematical books.

As we entered the new millennium mathematicians decided to repeat Hilbert's challenge. In issue 24 of *Plus*, we saw How maths can make you rich and famous − by solving one of the seven Millennium Prize Problems. The Riemann Hypothesis is the only problem from Hilbert's list that is also on this new list. So read on − this article might be your passport to becoming a millionaire.

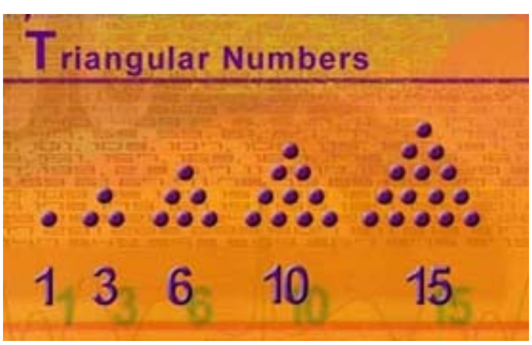
The Pattern searcher

One of the great pattern searchers: Leonardo Fibonacci

The Riemann Hypothesis goes to the heart of what it means to be a mathematician: looking for patterns. The search for patterns is perfectly captured by those challenges that invariably come up in the classroom: find the next number.

Here are a list of challenges for you to have a go at. Can you find the patterns and fill in the next numbers?

1	3	6	10	15				?
1	1	2	3	5	8	13		?
6	19	29	32	34	39			?
2	3	5	7	11	13	17	19	?



The first two probably presented little problem. The first sequence is known as the *triangular numbers*. The Nth number in the sequence records the number of stones in a triangle with N rows or in other words, the sum of the first N numbers: $1+2+...+N$.

Nature's numbers. Image DHD Photo Gallery

The second sequence is one of Nature's favourite sequences. Called the *Fibonacci numbers* after the thirteenth century mathematician who first recognized their importance, each number in the sequence is got by adding together the previous two numbers. Invariably the number of petals on a flower is a number in this sequence.

Not so easy to guess. Image DHD Photo Gallery

The third sequence was probably a little more challenging. Indeed, if you were able to predict that 46 was the next number in this sequence, I would recommend you buy a lottery ticket next Saturday. These were the winning lottery numbers on the 22 October 2003.

The last sequence is of course the sequence of *prime numbers*, the indivisible numbers that can only be divided by themselves and one. It is trying to explain the sequence of prime numbers that the Riemann Hypothesis is all about.

Faced with a sequence of numbers like these, numerous questions spring to the mathematical mind. As well as the challenge of finding the rule to predict the next number, mathematicians are also keen to understand if there is some underlying formula which can help generate these numbers: Is there a way to produce the 100th number on this list without having to calculate the previous 99?

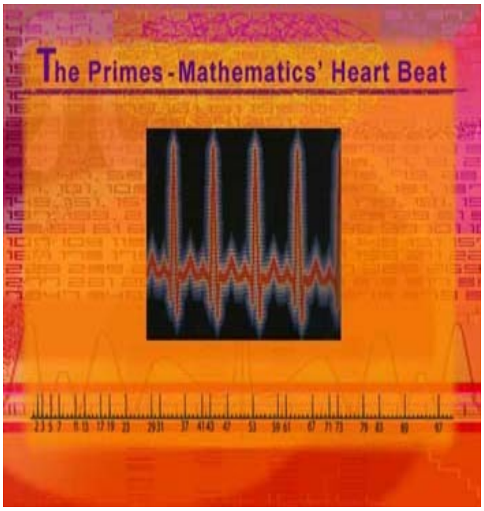
In the case of our sequences, the first two do indeed have formulas that will generate the sequence. For example, to get the 100th triangular number you just have to set N=100 in the formula $1/2 \times N \times (N+1)$. A much simpler task than adding up all the numbers from 1 to 100. (A challenge: can you prove why this formula will always work? Here's a hint: take two triangles and build a rectangle with N+1 rows and N columns.)

But when you look at the sequence of prime numbers they seem to share more in common with the numbers from the National Lottery. It seems very difficult to predict when the next prime will appear, let alone produce a formula that will tell you that the 100th prime is 541.

Despite the random nature of the primes they have a universal character. The National Lottery numbers have nothing special about them and change from one week to the next. The primes are numbers that have been there for eternity, set into the fabric of the universe. The fact that 541 is prime is a fact that seems stamped into the nature of the Universe. There maybe a different chemistry or biology on the other side of the cosmos but 541 will still be prime. This is why many science fiction writers (for example, Carl Sagan, in his classic novel, *Contact*, also made into a film of the same name) have chosen primes as the way alien life will communicate with Earth. There is something so special about this sequence of numbers that we couldn't fail to notice the prime number beat if it came pulsating through the cosmos from a distant galaxy.

Aliens may have discovered the primes millions of years ago, but what is the first evidence of humankind listening to the prime number beat? Some people have suggested that the first culture to recognize the primes lived over 8,000 years ago. Archaeologists discovered a bone now called the *Ishango bone* in Central Equatorial Africa, which has four columns of notches carved down the side. The bone seems to be mathematical in nature. And in one column we find the primes between 10 and 20. But others have argued that these bones are related to keeping track of dates and it is the random nature of the primes which means that a list of dates could well be a list of primes. (You can find out more about the Ishango bone at Ishango: The exhibition.)

The first culture to truly understand the significance of the primes to the whole of mathematics was the Ancient Greeks. They realised that the primes are the building blocks of all numbers. Every number can be built by multiplying together prime numbers. They are the atoms of arithmetic. Every subject has its building blocks: Chemistry has the Periodic Table, a list of 109 elements which build matter; physicists have the fundamental particles which range over weird things as quarks and gluons; biology is seeking to sequence the human genome which is the building kit for life.



But for thousands of years mathematicians have been listening to the primes, the heart beat of mathematics, unable to make sense or to predict when the next beat will come. It seems a subject wired by a powerful caffeine cocktail. It is the ultimate tease for the mathematician: mathematics is a subject of patterns and order and symmetry, yet it is built out of a set of numbers that appears to have no rhyme or reason to them. For two thousand years we have battled to understand how Nature chose the atoms of arithmetic.

There is of course the possibility that, as with the atoms of chemistry, there are only 109 primes which can be used to build all numbers. If that were true, we wouldn't need to worry about looking for patterns to predict primes because we could just make a finite list of them and be done. The great Greek mathematician Euclid scuppered that possibility. In what many regard as the first great theorem of mathematics, Euclid explained why there are infinitely many prime numbers. This proof is described in A whirlpool of numbers from Issue 25 of *Plus*. Gone, then, is the chance to produce a list containing all the prime numbers like a Periodic Table of primes or prime genome project. Instead we must bring our mathematical tools to bear to try to understand any patterns or structure in this infinite list.

Perhaps the primes start off rather unpredictably and then settle down into a pattern. Let's have a look at the primes around 10,000,000 and see whether here a pattern emerges, whether we might find a formula for predicting the ebb and flow of the primes.

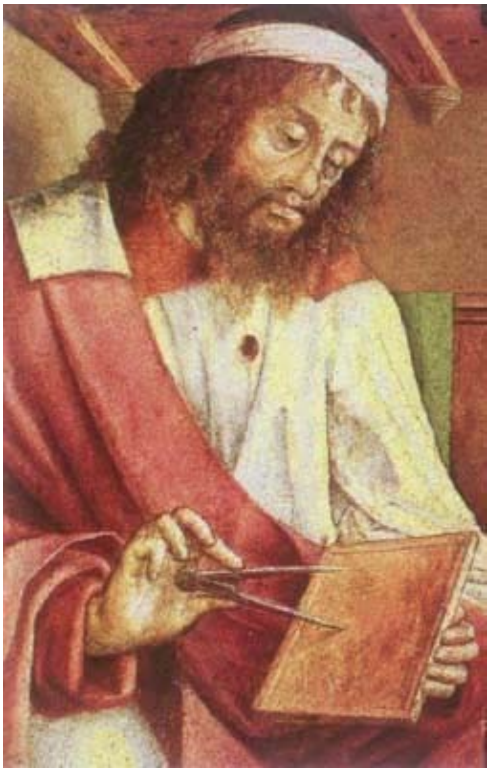
There are 9 primes in the 100 numbers before 10,000,000:

- 9,999,901,
- 9,999,907,
- 9,999,929,
- 9,999,931,
- 9,999,937,
- 9,999,943,
- 9,999,971,
- 9,999,973,
- 9,999,991.

But look how few there suddenly are in the hundred numbers after 10,000,000:

- 10,000,019,
- 10,000,079.

The primes seem to come along like buses: first a great cluster of primes and then you have to wait for ages before the next ones appear. It seems hopeless to find a formula that would spit out this strange list or that would tell us that the 664,571th prime is 9,999,901.



Euclid, who discovered that there are infinitely many primes

Try a little experiment. Take the list of primes around 10,000,000 and try to memorize them. Switch off the computer and see how well you do at reproducing the list. Most of us will try to create some underlying pattern to help us memorize the sequence. Effectively our brains end up trying to store a shorter program that will create the sequence. A good measure of the random character of these numbers is the fact our brains find it hard to construct a program that is significantly shorter than just memorizing the sequence outright.

It's like the difference between listening to a tune and listening to white noise. The inner logic of the tune allows you to whistle it back after a few times, while the white noise gives you no clues as to where to whistle next. The magic of the primes is that, despite first hearing only white noise, a cultural shift into another area of mathematics will reveal an unexpected harmony. This was Gauss's and Riemann's great insight. Like western ears listening to the music of the east, we will require a different perspective before we understand the pattern responsible for this randomness.

The lateral thinker

What mathematicians are good at is lateral thinking. Professor Enrico Bombieri, a professor in Princeton, expresses what one should do when facing an insurmountable mountain: "When things get too complicated, it sometimes makes sense to stop and wonder: Have I asked the right question?"

It was a fifteen year old boy called Carl Friedrich Gauss who started to change the question. Gauss became an overnight star in the scientific community at the beginning of the nineteenth century thanks to a tiny tumbling rock. The first day of the new century had started auspiciously with the discovery of what many regarded as a new planet between Mars and Jupiter. Christened Ceres, its path was tracked for several weeks when suddenly astronomers lost it as it passed behind the sun. But Gauss was up to the challenge of finding some order in the data that had been collected. He pointed at a region of the sky that astronomers might be expected to see Ceres. Sure enough, there it was.

Carl Friedrich Gauss, 1803

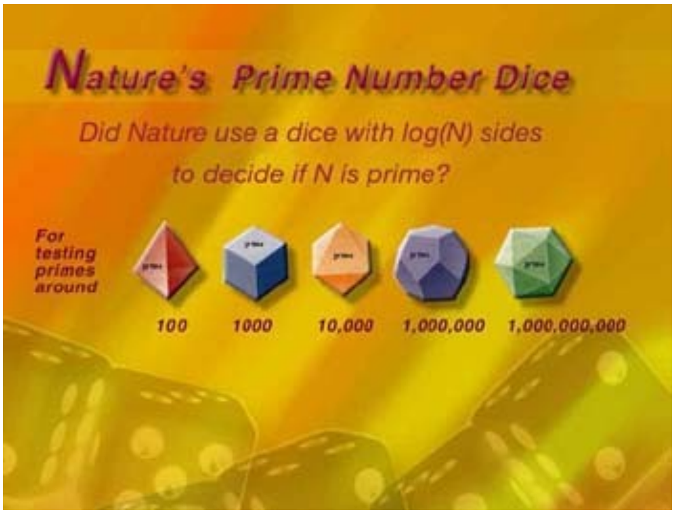
But it wasn't only in the stars that Gauss loved to find order. Numbers were his passion. And of all the numbers that fascinated Gauss, the primes above all were the jewels that he wanted to understand. He had been given a book of logarithms as a child which contained in the back a table of primes. It is somewhat uncanny that the present contained both, because Gauss managed to find a connection between the two.

Gauss decided he would see whether he could count how many primes there were rather than just trying to predict which numbers were prime. Here was the lateral move that would eventually unlock the secret of the primes. He asked: What proportion of numbers are prime numbers? He discovered that primes seemed to get rarer as one counted higher. He made a table recording the changing proportions.

N	**Number of primes from 1 up to *N*, often referred to as Π(*N*)**	**On average, how many numbers you need to count before you expect a prime number**
10	4	2.5
100	25	4.0
1,000	168	6.0
10,000	1,229	8.1
100,000	9,592	10.4
1,000,000	78,498	12.7
10,000,000	664,579	15.0
100,000,000	5,761,455	17.4
1,000,000,000	50,847,534	19.7

10,000,000,000	455,052,511	22.0

So, for example, 1 in 6 numbers around 1,000 are prime.



Since the primes look so random, perhaps tossing a dice might provide a good model of how the primes were distributed. Maybe Nature used a prime number dice to choose primes around 1,000, "PRIME" written on one side and the other five sides blank. To decide if 1,000 was prime, Nature tossed the dice to see if it landed on the "PRIME" side. Of course this is just a heuristic model. A number is prime or it isn't. But Gauss believed that this "prime number dice" might produce a list of numbers with very similar properties to the true list of primes.

How many sides are on the dice as we check the primality of bigger and bigger numbers? For primes around 1,000, Nature appears to have used a six−sided dice; for primes around 10,000,000 one needs a 15−sided−dice. (So there is a 1 in 15 chance that a London telephone number is prime.) Gauss discovered that the tables of logarithms at the beginning of his book containing the tables of primes provided the answer to determining how many sides there are on the prime number dice.

Look again at Gauss's table counting the number of primes. Whenever Gauss multiplied the first column by 10, then the last column, recoding the number of sides on the prime number dice, goes up by *adding* approximately 2.3 each time. Here was the first evidence of some pattern in the primes. Gauss knew another function which performed the same trick of turning multiplication into addition: the logarithm function.

It was a 17th century Scottish baron, John Napier, who first discovered the power of the logarithm as an important function in mathematics. Napier was generally regarded to be in league with the devil as he strode round his castle with a black crow on his shoulder and a spider in a little cage muttering about the predictions of his apocalyptic algebra. But he is best remembered today for the invention of the logarithm function.

If you input a number N into the logarithm function, it outputs a number $x=\log_{10}(N)$ which solves the following equation:

$$10^x = N.$$

So, for example,

$$\log_{10}(10)=1;$$

$$\log_{10}(100)=2;$$

$$\log_{10}(1{,}000{,}000)=6.$$

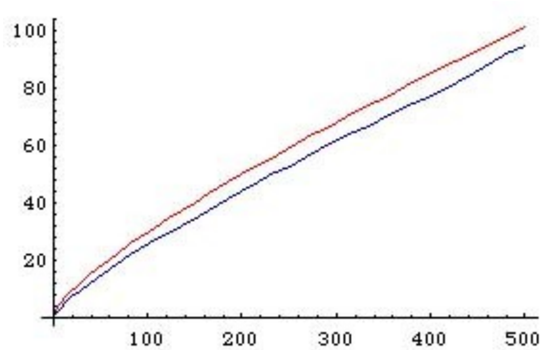
Whenever I multiply the input by 10 then I add 1 to the output.

But we needn't choose the number 10 to raise to the power x. That's just our obsession with the fact that we have ten fingers. Choosing different numbers gives logarithms to different bases. Gauss's prime number dice function goes up by 2.3 every time I multiply by 10. The logarithm behind this function is to the base of a special number called e=2.718281828459... .

Gauss guessed that the probability that a number N is prime is $1/\log(N)$ where log is taken to the base e. This is the probability that a die with $\log(N)$ sides lands on the "PRIME" side. Notice that as N gets bigger, $\log(N)$ gets bigger and the chance of landing on the prime side gets smaller. Primes get rarer as we count higher.

If Nature tosses the prime number dice 100,000 times, how many primes will you expect to get with these dice with varying numbers of sides? If the die has a fixed number of sides, say 6, then you expect 100,000/6 which is the probability 1/6 added up 100,000 times. Now Gauss is varying the number of sides on the die at each throw. The resulting number of primes is expected to be

$$1/\log(2)+1/\log(3)+...+1/\log(100{,}000).$$



The staircase of the primes versus Gauss's guess Li(N)

Gauss refined this guess at the number of primes into a function called the *logarithmic integral,* denoted by Li(N). How good is Gauss's guess compared to the real number of primes? We can see the difference in the graph to the left. The red line is Gauss's guess using his prime number dice; the blue one records the real number of primes.

Gauss's guess is not spot−on. But how good is it as we count higher? The best measure of how good it is doing is to record the percentage error: look at the difference between Gauss's prediction for the number of primes and the true number of primes as a percentage of the true number of primes.

N	**Number of primes $\Pi(N)$ from 1 up to N.**	**How far Gauss's guess Li(N) overestimates the number of primes**	**Percentage error: $[(\mathrm{Li}(N)-\Pi(N))/(\Pi(N))]\times 100$**

		less than *N*: Li(*N*)− Π(*N*)	
100	25	5	20
1,000	168	10	5.95
10,000	1,229	17	1.38
100,000	9,592	38	0.396
1,000,000	78,498	130	0.166
10^7	664,579	339	0.051
10^8	5,761,455	754	0.0131
10^9	50,847,534	1,701	0.00335
10^{10}	455,052,511	3,104	0.000682
10^{11}	4,118,054,813	11,588	0.000281
10^{12}	37,607,912,018	38,263	0.000102
10^{13}	346,065,536,839	108,971	0.0000299
10^{14}	3,204,941,750,802	314,890	0.00000983
10^{15}	29,844,570,422,669	1,052,619	0.00000353
10^{16}	279,238,341,033,925	3,214,632	0.00000115

Gauss believed that as we counted higher and higher the percentage error would get smaller and smaller. He didn't believe there were any nasty surprises waiting for us. His belief became known as:

> *Gauss's Prime Number Conjecture: The percentage error gets smaller and smaller the further you count.*

There is a lot of evidence here, but how can we really be sure that nothing weird happens for very large *N*?

In 1896 Charles de la Vallée−Poussin, a Belgian, and Jacques Hadamard, a Frenchman, proved that Gauss was correct. But here is a warning that it is far from obvious that patterns will persist. Gauss also thought that his guess would always *overestimate* the number of primes. The evidence from the tables looks overwhelming. But in 1912 Littlewood, a mathematician in Cambridge, proved Gauss was wrong. However the first time that Gauss's guess underestimates the primes is for *N* bigger than the number of atoms in the observable universe − not a fact that experiment will ever reveal.

Georg Friedrich Bernhard Riemann

Gauss had discovered the "Prime Number Dice" used by Nature to choose the primes. They were dice whose number of sides increased as larger and larger primes are chosen. The number of sides grew like the logarithm function. The problem now was to determine how this dice landed. Just as a coin rarely lands exactly half heads and half tails, Gauss still didn't know exactly how this dice was landing.

It was Gauss's student Riemann who discovered that music gave the best explanation of how to get from the graph showing Gauss's guess to the true graph of the number of primes. As we shall discover in next issue's instalment, Riemann's music could explain how Nature's dice really landed.

About this article

Marcus du Sautoy

Professor Marcus du Sautoy is an Arsenal fan. His favourite primes are featured on the cover of his book *The music of the primes* (reviewed in last issue of *Plus*).

There is the chance to explore more about the exciting world of prime numbers in the interactive website www.musicoftheprimes.com. You can build a prime number fantasy football team and get it to play other teams. Will you be top of the prime number premiership?

And why not take the opportunity to do an experiment with prime number cicadas? By choosing different life cycles for the cicada and the predator, the game shows why primes became the key to the cicadas survival.

There is also the chance to contribute to the Prime Number Photo Gallery. Do you have a picture of your favourite prime number? Have you seen a prime number somewhere strange − like the date on a building? Can you find pictures for the missing primes in our gallery?

And there are more details about how to win a million dollars and prove the Riemann Hypothesis.

Download PDF version | Printer friendly version

Return to article

- contact
- copyright info
- sponsors
- privacy info