September 1997
Regulars

Solution to the optimal stopping problem

The probability of choosing the best partner when you look at $M-1$ out of N potential partners before starting to choose one will depend on M and N. We write $P(M,N)$ to be the probability.

For any value of N, this probability increases as M does, up to a largest value, and then falls again. $P(1,N)$ and $P(N,N)$ will always be $1/N$ because these two strategies, picking the first or last potential partner respectively, leave you no choice: it's just like picking one at random. Our aim is to find when $P(M,N)$ is largest.

Suppose that you have collected the information from $M-1$ potential partners and are considering the Kth in sequence.

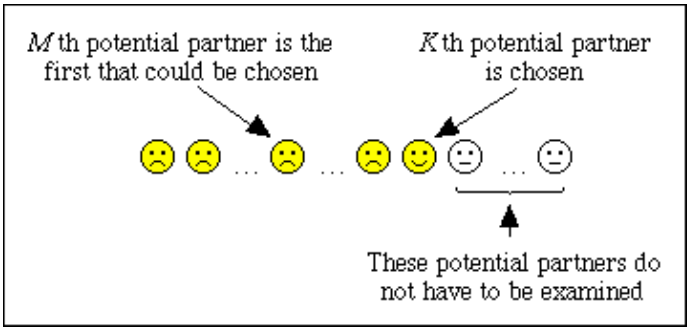
$$M-1 < K \leq N$$

Since the potential partners come along in a random order, the chance that this one is the best is $1/N$. But you only consider this potential partner if the highest ranking potential partner that you've seen so far was among the first $M-1$ of the $K-1$ that you have rejected (otherwise you wouldn't be looking at this potential partner at all). This happens with the following probability:

$$\frac{M-1}{K-1}$$

So the overall chance of achieving your aim of finding the best potential partner this time is:

$$\frac{M-1}{N(K-1)}$$



But *K* can take any of the values in the range from *M* to *N*, so we can write:

$$P(M,N) = \sum_{K=M}^{N} \frac{M-1}{N(K-1)} = \frac{M-1}{N} \sum_{K=M}^{N} \frac{1}{K-1}$$

The best value of M will be the one which satisfies:

$$P(M-1,N) < P(M,N) < P(M+1,N)$$

(If you want to be very awkward, you could ask what happens if there are two "best" values of *M*, with one of those strict inequality signs replaced by a partial inequality. It turns out that the only time when equality is possible is when *N*=2, which is not very interesting anyway.)

Taking the first inequality, $P(M-1,N) < P(M,N)$:

$$\frac{M-2}{N} \sum_{K=M-1}^{N} \frac{1}{K-1} < \frac{M-1}{N} \sum_{K=M}^{N} \frac{1}{K-1}$$

Removing the redundant factor of $1/N$ and rewriting the LHS:

$$(M-2)\left(\frac{1}{M-2} + \sum_{K=M}^{N} \frac{1}{K-1}\right) < M-1 \sum_{K=M}^{N} \frac{1}{K-1}$$

$$1 < \sum_{K=M}^{N} \frac{1}{K-1}$$

Similarly the second inequality, $P(M,N) > P(M+1,N)$ gives:

$$M-1 \sum_{K=M}^{N} \frac{1}{K-1} > M \sum_{K=M+1}^{N} \frac{1}{K-1}$$

$$1 + M-1 \sum_{K=M+1}^{N} \frac{1}{K-1} > M \sum_{K=M+1}^{N} \frac{1}{K-1}$$

$$1 > \sum_{K=M+1}^{N} \frac{1}{K-1}$$

We can use these inequalities to find *M* for any *N*. Try it! Fill in the blanks below:

$$\frac{1}{M}+\frac{1}{M+1}+\ldots+\frac{1}{N-1}<1<\frac{1}{M-1}+\frac{1}{M}+\ldots+\frac{1}{N-1}$$

N	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
M	2	2		3	3	3	4		5	5	6	6	6		7	7	8	8

[Answers]

The fraction of the potential partners that you see M/N is tending to a limit as N becomes large. There is a sum in the calculation of $P(M,N)$ which appears in other situations in mathematics too:

$$\sum_{K=1}^{L}\frac{1}{K}\cong\ln(L)+C \qquad \text{Where } C \text{ is "Euler's constant"}$$

Using this equation we can calculate an approximation for $P(M,N)$ as follows:

$$\begin{aligned}
P(M,N) &= \frac{M-1}{N}\sum_{K=M}^{N}\frac{1}{K-1}=\frac{M-1}{N}\sum_{K=M-1}^{N-1}\frac{1}{K}\\
&= \frac{M-1}{N}\left\{\sum_{K=1}^{N-1}\frac{1}{K}-\sum_{K=1}^{M-2}\frac{1}{K}\right\}\\
&\cong \frac{M-1}{N}\{\ln(N-1)-\ln(M-2)\}\\
&\cong \frac{M-1}{N}\ln\left(\frac{N-1}{M-2}\right)
\end{aligned}$$

For big N, we can make it even more simple:

$$P(M,N)\cong\frac{M}{N}\ln\left(\frac{N}{M}\right)$$

In order to find the *best* value of M we have to apply the approximation to the conditions that we derived before:

$$1<\sum_{K=M}^{N}\frac{1}{K-1}=\sum_{K=M-1}^{N-1}\frac{1}{K}=\sum_{K=1}^{N-1}\frac{1}{K}-\sum_{K=1}^{M-2}\frac{1}{K}\cong\ln\left(\frac{N-1}{M-2}\right)$$

$$1>\sum_{K=M+1}^{N}\frac{1}{K-1}=\sum_{K=M}^{N-1}\frac{1}{K}=\sum_{K=1}^{N-1}\frac{1}{K}-\sum_{K=1}^{M-1}\frac{1}{K}\cong\ln\left(\frac{N-1}{M-1}\right)$$

Therefore, for large N, the best M satisfies:

$$1\cong\ln\left(\frac{N}{M}\right)$$

$$\frac{M}{N}\cong\frac{1}{e}$$

$1/e$ is about 0.368. This result can be expressed simply in the following "37%" rule:

37% rule

Look at a fraction $1/e$ of the potential partners before making your choice and you'll have a $1/e$ chance of finding the best one!

Further reading:

For more information see the article "Mathematics, marriage and finding somewhere to eat" elsewhere in this issue.

Answers to problems:

N	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
M	2	2	3	3	3	3	4	4	5	5	6	6	6	7	7	7	8	8

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