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Regulars

Solution to Puzzle No. 3 − birth dates

> If X_N appears as *XN* then your browser does not support subscripts or superscripts. Please use this alternative version.

For the question see "Puzzle No. 3 − birth dates", in issue 3.

The answer to the problem is that, in order to share this special numerical relationship, the age of the mother must be a multiple of 9 when the child is born.

Why?

To prove that this is the case, we need a little theorem.

Theorem

If N is a positive integer and S is the sum of its digits, then N mod 9 = S mod 9.

"N mod 9" just means "the remainder r when N is divided by 9", where r can range from 0 to 8.

Proof

The proof, in a nutshell, looks like this:

If the digits of N are x_1, x_2,... , x_n then

$$N = 10^{n-1}x_1 + 10^{n-2}x_2 + \ldots + 10x_{n-1} + x_n$$

But it is always true that

$$10^m = \underbrace{99\ldots9}_{m-1\text{ times}} + 1$$

So we can write:

$$N = \underbrace{99\ldots9}_{n-2}x_1 + \underbrace{99\ldots9}_{n-3}x_2 + \ldots + 9x_{n-1} + (x_1 + x_2 + \ldots + x_n)$$

$$N = 9(\underbrace{11\ldots1}_{n-2}x_1 + \underbrace{11\ldots1}_{n-3}x_2 + \ldots + x_{n-1}) + S$$

But the first term of this expression is an exact multiple of 9.
Therefore N mod 9 = S mod 9.

QED.

This theorem tells us that summing the digits of a number does not change its value mod 9. Therefore, repeatedly summing the digits of a number until a single digit is reached does not change its value mod 9.

On the day a child is born its age is 0. Therefore, to share this special numerical relationship with its parent the parent's age mod 9 must also be 0. This is simply another way of saying that the parent's age must be a multiple of 9, e.g., 9, 18, 27, 36, 45, 54,...

Perhaps a more famous use of this theorem is in deducing that a number is divisible by 3 if and only if the sum of its digits is divisible by three. Why?

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.