January 1998
Regulars

Solution to the coin tossing problem

BOOKMARK

> If X_N appears as *XN* then your browser does not support subscripts or superscripts. Please use this alternative version.

The chance that any fixed set of *k*+1 tosses gives *k* heads followed by a tail is:

$$\left(\frac{1}{2}\right)^{k+1}$$

In *n* tosses there are *n*−*k* possible starting points of such sequences. Therefore the expected number of wins is:

$$(n-k)\left(\frac{1}{2}\right)^{k+1}$$

Using probability theory, one finds that the actual number of wins has what is called an *approximate Poisson distribution*. That is to say, the probability that you win exactly *r* pennies is given approximately by the *Poisson probability*:

$$p_r = \frac{\lambda^r}{r!}e^{-\lambda}$$

where λ is the Greek letter *lambda* and

$$\lambda = (n-k)\left(\frac{1}{2}\right)^{k+1}$$

Your average number of wins equals *lambda*. If *lambda* is big, then the number of wins will generally be big also, and if *lambda* is small then there will be only a small number of wins (perhaps 0).

The threshold between big and small occurs when *lambda* is a 'reasonable' number, say the value 1. When *lambda* = 1, then *k* is very close to $\log_2(n)$. In this threshold case the Poisson probabilities are given in the following table.

p_0	p_1	p_2	p_3	p_4	p_5
0.368	0.368	0.184	0.061	0.015	0.003

Substantially longer runs than $\log_2(n)$ are exceedingly unlikely, while substantially shorter runs are commonplace.

Acknowledgements

This solution was written by Geoffrey Grimmett. You may also be interested in his article "What a coincidence!" elsewhere in this issue.

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.