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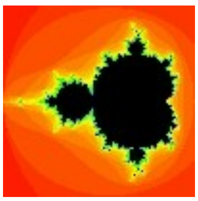
Features



Unveiling the Mandelbrot set

by Robert L. Devaney

BOOKMARK

The filled Julia set of $x^2 + 0$

To work out the filled Julia set of $x^2 + 0 = x^2$ we need to look at all the points on the complex plane and decide whether or not their orbits escape to infinity. Those whose orbit doesn't escape are part of the filled Julia set.

Let's start with a complex number $a + ib$ and let's assume that its distance from the complex number 0 is greater than 1. On the plane, the number $a + ib$ is represented by the point with co−ordinates (a,b) and 0 is the point with co−ordinates $(0,0)$.
This means that $a + ib$ has distance $\sqrt{a^2+b^2}$ from 0, and so, by our assumption, we have

$$\sqrt{a^2+b^2} > 1.$$

Now using $a + ib$ as the seed x_0 we get

$$x_1 = (a+ib)^2 = a^2 - b^2 + i2ab,$$

So the distance of x_1 to 0 is

$$\sqrt{(a^2-b^2)^2 + 4a^2b^2}.$$

So is x_1 closer to 0 than x_0 or further away? Well, we have

$$(a^2-b^2)^2 + 4a^2b^2 \;=\; a^4 - 2a^2b^2 + b^4 + 4a^2b^2 \;=\; a^4 + 2a^2b^2 + b^4 \;=\; (a^2+b^2)^2.$$

Hence, the distance of x_1 to 0 is $a^2 + b^2$. But since $\sqrt{a^2 + b^2} > 1$ we know that $\sqrt{a^2 + b^2} < a^2 + b^2$, and so x_1 is further away from 0 than x_0. Repeating this argument shows that x_2 is further away from 0 than x_1, x_3 is further away from 0 than x_2, and so forth. In other words, points in the orbit of x_0 move further and further out the orbit tends to infinity. This means that x_0 does not lie in the filled Julia set. And since x_0 represented *any* point with distance greater than 1 from 0, we know that no such point can lie in the filled Julia set.

A very similar calculation shows that if the distance between $x_0 = a + ib$ and 0 is less than 1, then x_1 is closer to 0 than x_0, and this means that the orbit cannot possibly tend to infinity. So points whose distance is to 0 is less than 1 lie in the filled Julia set.

And what if the distance between $x_0 = a + ib$ and 0 is equal to 1? A calculation shows that the distance between x_1 and 0 is also equal to 1. Thus, points in the orbit of x_0 always remain at distance 1 from 0 the orbit does not tend to infinity and therefore x_0 lies in the filled Julia set.

We've now accounted for all the points on the plane and seen that only those whose distance to 0 is less than or equal to 1 belong to the filled Julia set. The filled Julia set, then, is the disc in the plane with centre 0 and radius 1.

If you are familiar with complex numbers the calculation is actually easier. In this case you will know that a complex number $x_0 = x + iy$ can be written as $x_0 = re^i$, where r is its distance to the point 0 and is the angle that the line from 0 to x_0 makes with the x−axis. Now $x_1 = r^2e^{2i}$, so x_1 has distance r^2 from 0. This means that x_1 is further a way from 0 than x_0 precisely when $r > 1$.

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