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Features

Non−Euclidean geometry and Indra's pearls

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A brief introduction to complex numbers

Complex numbers are based on the number i which is defined to be the square root of −1, so i times i equals −1. This number isn't a real number, in other words it does not appear on the usual number line. For this reason it is called an *imaginary number*, a slightly contentious name. Now any complex number is of the form $a + ib$, where a and b are ordinary real numbers. The numbers $1 + i2$ or $5 - i8$ are both complex numbers.

Unless a complex number $a + ib$ has $b = 0$, we cannot find it on the ordinary number line. We can, however, visualise it as a point on the plane: to the number $a + ib$ simply associate the point with co−ordinates (a,b). You can see that the real numbers are contained in the complex numbers: a real number x seen as a complex number is simply $x + i0$ and corresponds to the point with co−ordinates $(x,0)$.

Complex numbers are added (as you would expect) like this:

$$(a + ib) + (c + id) = (a + c) + i(b + d),$$

and multiplied (again as you would expect) like this:

$$(a + ib)(c + id) = ac + iad + ibc + i2bd = ac - bd + i(ad + bc).$$

Division of complex numbers is a little harder. Suppose that $z = x + iy$ and that $w = u + iv$ are two complex numbers. Then we calculate z/w by getting rid of the imaginary part in the denominator:

$(x + iy)/(u + iv) = ((x + iy)(u - iv))/((u + iv)(u - iv)) = (xu + yv + i(yu + xv))/(u^2 + v^2)$. Since the denomninator of this expression is a real number, we now have a new complex number $(xu + yv)/(u^2 + v^2) + i(yu + xv)/(u^2 + v^2)$.

Now suppose that you have a Möbius transformation of the form

$$z \to (az+b)/(cz+d),$$

where a, b, c and d are some fixed complex numbers. Using the rules for addition, multiplication and division above you can calculate its value, which is a new complex number, or, going back to our visual interpretation, a new point on the plane.

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