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Features

A risky business: how to price derivatives

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BOOKMARK

Black−Scholes in the limit

This section contains some undergraduate level mathematics.

The Binomial model is a very simple model for understanding the ideas behind option pricing. However, so far the stock price can only take finitely many values and furthermore can only move at discrete time points. Both of these features are somewhat undesirable, and so in order to get around this we will look at the limit as the number N of time periods tends to infinity. This will give us the celebrated Black−Scholes formula.

Firstly, we will adjust the expression $1+r$ for the value of the bond at time T to e^{rT}. This isnâ€™t an obvious thing to do − it is itself the result of a limiting process.

See *Plus* article Have we caught your interest for more details.

Before taking our limit, it is important that we choose our parameters so that they scale in the correct way. First of all, we fix a terminal time, T, which is the expiry time of the option. Then set our δt to be given by $\delta t = T/N$. We will assume that:

$$\begin{aligned} u &= e^{\sigma\sqrt{\delta t}+\mu\delta t} \\ d &= e^{-\sigma\sqrt{\delta t}+\mu\delta t}. \end{aligned}$$

Furthermore, we assume that the probability of a good or bad period is the same,that is the probability of either period is 1/2 (although as before, this probability wonâ€™t matter for the option pricing). The stock price at time T is:

$$S_t = S_0 e^{\mu T + \sigma\sqrt{\delta t}(\xi_1 + \xi_2 + ... + \xi_N)}, \quad (1)$$

where ξ_i is a random variable which is 1 if the ith period was good and −1 otherwise. Now, we need to work out the equivalent probability, q, of an up jump, just as we did before. We have that:

$$q = \frac{e^{r\delta t} - u}{u - d}.$$

If we use a Taylor expansion on the exponential terms, we see that this gives:

$$q = \frac{1}{2}(1 - \sqrt{\delta t}\frac{\mu - r + \sigma^2/2}{\sigma}) + O(\delta t).$$

We want to look at the limit as N tends to infinity. From our earlier analysis, we should base the price of the option on the behaviour of equation 1, but assuming that the probability of ξ being +1 is given by q, rather than 1/2. The Central Limit Theorem tells us that if we have a sum of independent and identically distributed variables, which we sum and scale in the correct way, then in the limit we obtain a Normal distribution. To apply the CLT, we must first calculate the expectation of the random terms in the exponential in equation 1. Remember that we must use the probabilities q to calculate this expectation:

$$E^Q[\sigma\sqrt{\delta t}(\xi_1 + \xi_2 + ... + \xi_N)] = \sigma\sqrt{\delta t}(2q - 1) = -(\mu - r + \sigma^2/2)T,$$

where we have ignored the $O(\delta t)$ terms. We have can then apply the Central Limit Theorem:

$$\sigma\sqrt{\delta t}(\xi_1 + \xi_2 + ... + \xi_N) = \sigma\sqrt{T}\frac{(\xi_1 + \xi_2 + ... + \xi_N)}{\sqrt{N}} \rightarrow_{N\rightarrow\infty} -(\mu - r + \sigma^2/2)T + \sigma\sqrt{T}N(0,1),$$

where the convergence is in distribution and $N(0,1)$ denotes a Normal distribution with mean 0 and variance 1. So we see that in the limit, we should price based on the assumption that:

$$S_t = S_0 e^{\mu T - (\mu - r + \sigma^2/2)T + \sigma\sqrt{T}N(0,1)} = S_0 e^{\sigma\sqrt{T}N(0,1) + (r - \sigma^2/2)T}.$$

Note that the true drift of the stock, μ, does not matter for pricing these options, since the value of μ cancels in the expressions above.

We know now that we should base pricing on the formula given above. Therefore, we can now go about pricing the call option mentioned above. The price of this call option is:

$$E[(S_T - K)_+] \quad = \quad E[(S_0 e^{\sigma\sqrt{T}N(0,1) + (r - \sigma^2/2)T} - K)_+]$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} (S_0 e^{\sigma\sqrt{T}x+(r-\sigma^2/2)T} - K)_+ dx$$

$$= \int_{a}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} (S_0 e^{\sigma\sqrt{T}x+(r-\sigma^2/2)T} - K) dx$$

where a is the least value of x such that $(S_0 e^{\sigma\sqrt{T}x+(r-\sigma^2/2)T} - K) \geq 0$, that is:

$$a = \frac{\log(Ke^{-rT}/S_0) + \sigma^2/2}{\sigma\sqrt{T}}.$$

We may easily compute this integral (complete the square in the exponential) to obtain the price as:

$$S_0\bar{\Phi}(a - \sigma\sqrt{T}) - e^{-rT}K\bar{\Phi}(a),$$

where Φ is the cumulative distribution function of a standard Normal random variable and $\bar{\Phi}(a) = 1 - \Phi(a)$. Rearrangement gives the price as:

$$S_0\Phi(d_1) - e^{-rT}K\Phi(d_1 - \sigma\sqrt{T}),$$

where $d_1 = \frac{\log(e^{rT}S_0/K)+\sigma^2T/2}{\sigma\sqrt{T}}$. This is the famous Black−Scholes formula.

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