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Features

Chaos, chance and money

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The stock market: deterministic or random?

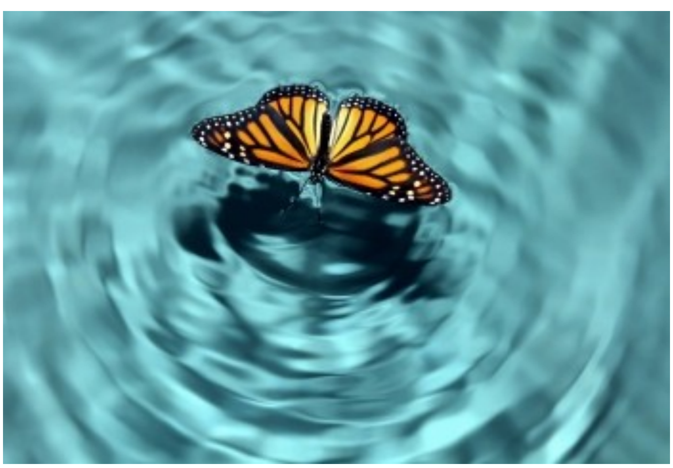
Recently, the credit crunch has dominated the news. Columnists wail about "chaos in the markets", and share prices look likely to remain "turbulent" for some time. We are nervously waiting to see whether the government's buyout of parts of the banks will return us to stability, whilst blaming the bankers for incorrectly pricing risk. I've been amused by the number of mathematical terms I've heard in the news, particularly as one cause of the credit crunch was borrowers not understanding the mathematical implications of their mortgage deals.

So what does move the markets? Are they deterministic dominated by the laws of cause and effect? Or are there chance factors involved, making it unlikely that the government can change their behaviour?

From order to chaos

Let's start by considering a completely deterministic system, with no room for chance: the world according to Sir Isaac Newton. Newton's laws of motion describe how objects behave under forces like gravity. They are phrased in the language of differential equations, which describe rates of change. The precision, accuracy and seemingly universal applicability of his laws made people imagine the universe as a giant clockwork machine. Given enough data describing the world right now, it should be possible to predict exactly what will happen in the future, forevermore. This raises serious philosophical problems for free will, but the idea of such accurate knowledge was seductive to scientists, and the clockwork universe dominated our world view for over 200 years.

However, we've all recently seen crazy movements in the stock market, with billions of pounds being lost in only a few hours. If the markets are working like clockwork, it's not a clock that you'd rely on! Over the past forty years, mathematicians have increasingly realised that deterministic behaviour can fluctuate so unpredictably that it looks random: the study of this is called *chaos theory*. And the first example of chaos came from Newton's very laws. When two stars orbit one another, we can predict their positions for millennia, but introduce a third and they start wobbling as unstably as the FTSE: this is the *three body problem,* and it produces chaos.



Far−reaching consequences.

But what exactly is chaos? One of its features is that tiny changes to the starting conditions can result in wildly different behaviour. In the 1960s the meteorologist Edward Lorenz was studying computer models of the atmosphere. To check for computer glitches, he would record the data from half−way through each calculation, then start the computer model again from there. One time, he was astonished to find dramatically different results after the restart. At first he thought it was a bug, but eventually he worked out what was going on: he had typed 0.506 the number on the computer printout instead of 0.506127 the number the machine had stored. These differ by only 1 part in 1000, but as the computer chomped through a huge number of calculations, this tiny difference amplified into a huge discrepancy. Lorenz christened this the *butterfly effect*, because this sensitivity of the weather to tiny changes means that a butterfly flapping its wings in China could theoretically cause a hurricane in the United States. This is why weather predictions only work for a few days: tiny errors rapidly swamp the correct forecast.

Incidentally, chaos is not always bad news. In 1990, the Japanese sent a small probe to the Moon, whilst a larger spacecraft orbited the earth. The small probe's radio failed, leaving no way to control it. The larger craft had only one tenth of the fuel required to reach the Moon, and the situation seemed hopeless. However, the Japanese team remembered that ten years before a mathematician called Edward Belbruno had shown that the three body problem can produce usable routes that require almost no fuel. The complex interaction of the planets and stars can cause gravitational forces to combine and form channels which propel anything in them to a certain destination. NASA hadn't been interested, as the chaotic routes were too slow, so the idea had been dropped. But the Japanese tracked Belbruno down, and he found a route to the Moon using only half of the remaining fuel! Thanks to the butterfly effect, the spacecraft got to the Moon and beyond. (See *Plus* article Lagrange and the interplanetary superhighway for more on these routes through space.)

Chaos in a deterministic world.

Another feature of chaotic systems is that they can suddenly flip from one behaviour to another, very different, one. Think about how some kinds of debt can suddenly become a risk rather than an asset to banks. Such a sudden change is known as a *phase transition*. One way to see a phase transition in action is to play with a tap. Turn it on very slowly and you hear a regular drip, drip, drip. Turn it up slightly and the drips become more frequent and not all the same size. A tiny bit more, and the drips become completely irregular and unpredictable: chaos sets in. Keep going, and the water settles down to a smooth column. However once you increase too much, the water starts twisting, frothing and spiralling: turbulence begins. By steadily increasing the flow we see regularity and chaos, without leaving the world of determinism.

Safety in numbers

So far we have considered the stock market as deterministic, and have seen that its swings in value imply that it's probably chaotic, and has recently gone through a phase transition. But what if the most important causes are due to chance? Isn't the real problem that none of us correctly identified our financial risks neither the borrowers taking out mortgages that they couldn't afford in the hope that house prices would go up, nor the banks buying up repackaged debts, gambling on quick returns?

'What, it's your birthday too?'
Fulham versus Bolton Wanderers. Image by D.Nutall.

To analyse chance, we need probability theory, which is a comparatively young branch of mathematics, at only 400 years old. Its youth is in part due to the fact that probabilistic concepts don't come naturally to us. As an example of how counter−intuitive probability can be, think of the well−known *birthday paradox*: how many people need to be in a room for the probability that two of them share a birthday to be at least 1/2? Thinking about all the possibilities, we feel like that ought to be a pretty large number, maybe 100. In fact, it's 23. Marcus du Sautoy pointed out to me that that's the same as the number of people in a football match: two teams of 11 and a referee. If you check the Premiership this weekend, and find the birthdays of all of the players and (tricky one!) the referee, you'll discover that around half the matches feature people who share a birthday. The surprise that we even professional mathematicians feel at the smallness of that number 23 shows how bad we are at probability.

Let's take another example: pensions. An annuity is what you buy with your pension pot a guaranteed annual amount of money for life. The history of annuities is littered with bankruptcies in the 17th century William III must have lost a fortune when he sold fixed−price annuities that paid out as much every year to someone who was 30 as someone who was 70, even though a 30−year−old will probably live much longer! Around that time, insurance scams became so popular that you could buy insurance against dying from drinking too much rum!

Insurers soon noticed that it's safer to insure lots of people at once, so that the few people who live unexpectedly long, and thereby put the insurer out of pocket, are balanced out by an equal number who die early. Underlying this is the *law of large numbers*: you have no way of predicting how long an individual will live, but once you look at a large group of people, you can be fairly confident that the proportion of those dying at certain ages will follow distributions you have observed in the past.

But isn't one of the causes of the financial crisis the bundling together of lots of debt? Yes, but there is a difference between combining things with the same risk, where the law of large numbers makes things quite predictable, and mixing up a whole load of different risks, where it can be hard to know anything about the end result.

From chaos to order

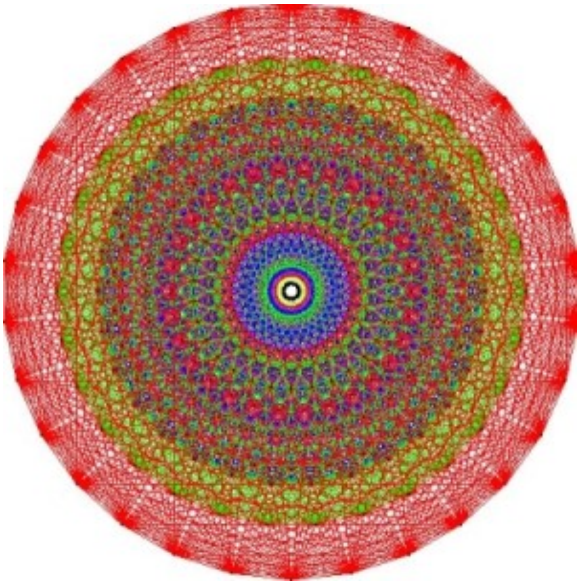
Sometimes, something even stronger than the law of large numbers applies, and random causes produce the same event, almost every time. I'd like you to imagine a set of 7 coloured coffee cups and saucers, one for each colour of the rainbow, arranged in a circle, in rainbow order: red, orange, yellow, and so on. Next to them is a piece of paper with an instruction. It says "move the cups two saucers clockwise". What patterns of cups and saucers can we reach by following the instruction repeatedly? The orange cup is next to the red cup, so it will always be its neighbour: if we move the red cup, we move the orange one too. So the cups stay in rainbow order. When we repeatedly move two along, the red cup goes past the orange saucer to the yellow one, then skips past the green for blue, then past indigo to violet, past red to orange, then to green and finally to indigo before returning to the red saucer. It visits every single saucer.

Actually, it doesn't matter whether the instruction says to move the cups two along, or any other number. In each case, the red cup will eventually reach every saucer, with all the other cups following along. So let's imagine that instead of one instruction on the table, there's a bag containing seven instructions: to move the cups 1 to 6 saucers along, or to leave them alone. We reach in and pick a random piece of paper. As long as we don't pick the instruction "don't touch the cups", if we follow it repeatedly, we eventually wind up with the same arrangements of cups and saucers: if we move the cups at all, the red cup eventually visits every saucer.

What makes this work is the fact that seven is a prime number. If the instruction "move the cups 2 saucers clockwise" took the red cup to only every other saucer, for example, then the number of cups would have to be even. But with a prime number of cups, as long as we don't pick "don't touch the cups", the red cup reaches all of the saucers. And as the number of cups gets bigger, our chances of picking "don't touch the cups" goes down: with 997 cups it's around 1 in 1000. And why did I pick the awkward number 997? Because it's prime!

The symmetry of chance

And this brings us to the notion of symmetry. Collections of symmetries with the property that following any one by any other gives one that was already there are called *groups*. This property that combining any two instructions gives one that is already there is called *closure*. Our coffee cup instructions are an example: "move two" then "move three" is the same as "move five". Groups are built from *simple groups*, which can't be broken down into smaller ones. The symmetries in our seven cups example form a simple group, because seven is prime. But with six cups I could have one bag telling me to move two or four along, or stay put, and another bag saying to move three along, or stay put. With only five instructions in two bags, each of which is a closed set of instructions, I can make all six instructions. The group of six symmetries can be broken up into two smaller sets, each a group in its own right.



Groups are a mathematical way of describing symmetry. This image is associated to the exceptional Lie group E8. It was generated on a computer by John Stembridge, based on a hand drawing by Peter McMullen.

Groups arise in many different contexts and there are infinitely many different ones. In fact, their basic building blocks, the simple groups, are also infinite in number. The simple groups can be as big as you like. One of the results that made me choose maths was the complete classification, in the 1980s, of the finite simple groups. This took thousands of pages of mathematics, done by hundreds of collaborating mathematicians: I wanted to join them! (See *Plus* article An enormous problem: the classification of finite simple groups.) Recently it was proved that if we pick two random symmetries from a finite simple group, and combine them in all possible ways, then as our choice of group gets bigger, the chance that we've made all of it gets closer and closer to 1. Complete predictability from random chance. One thing I do is design computer algorithms for groups I empathise with Lorenz and his problems with bugs! It turns out that randomised algorithms for groups are usually much faster than deterministic ones and the reason they work is because we can often prove that almost any random set of symmetries will produce the same behaviour.

I hope I've managed to convince you that seemingly predictable, regular behaviour does not necessarily have a deterministic cause it's perfectly possible for it to arise from chance events. Conversely, seemingly wild, erratic and downright chaotic behaviour can emerge from a system as simple as the interacting gravity of the Earth, the Moon, and a spaceship. The stock market displays both types of behaviour. The bad news for the government is that in some systems, like our table of coffee cups, almost all actions lead to the same result. The good news is that in chaotic systems, a very small action can change a huge and complex set of interacting behaviours. Fingers crossed that it works!

About this article

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Colva is a lecturer in Pure Mathematics at the University of St Andrews. She initially went to university to do a degree in English and Moral Philosophy, but got distracted along the way and wound up with a maths degree. Since then she has worked at Queen Mary, University of London and the University of Sydney. She is interested in finite groups, computational methods and artificial intelligence.

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.