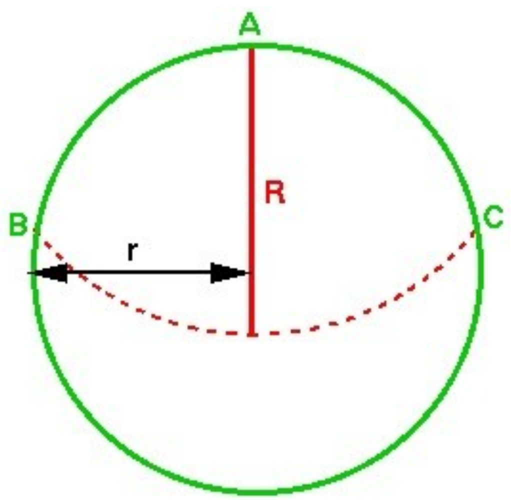

September 1999
Regulars

Solution to Puzzle No. 8

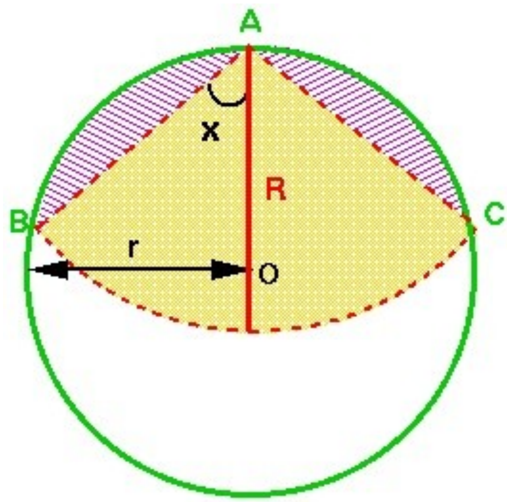
For the question see Puzzle No 8 − The Gobbling Goat in issue 8.



Let the circular field have radius r.

Let the length of the rope, which is anchored at point A on the circumference of the field, be R.

Now, with the rope at full stretch, the goat will be able to move in an arc from point B on the circumference to point C.



Let O be the centre of the field.

Clearly, the angle OAB is equal to the angle OAC. Let the magnitude of OAB be x radians.

Thus, the area accessible to the goat will be a circle sector with radius R and angle $2x$ (yellow), plus two circle segments (pink) from a circle of radius r, cut off by the chords AB and AC respectively.

Now, the area of the circle sector is:

$$\frac{1}{2}(R^2.2x) = R^2x$$

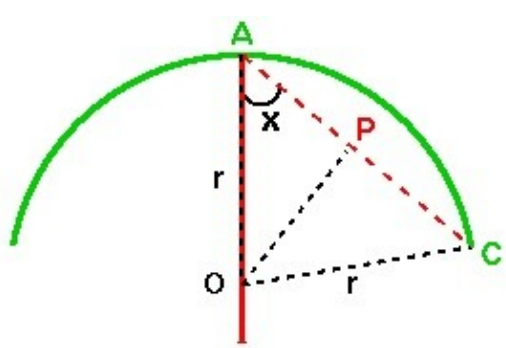
The area of each circle segment is:

$$(1/2)r^2(\pi - 2x) - (1/2)r^2\sin(\pi - 2x)$$

(because each is a sector of a circle minus a triangle) and so the total area accessible by the goat is:

$$R^2x + r^2[\pi - 2x - \sin(2x)]$$

(the yellow sector plus the two pink segments).



Currently, we have two different variables in our area equations: r and R. Letâ€™s try to eliminate one.

Obviously, the length of the line segment AO is r, the radius of the field. Similarly, the radius of the line segment OC must be r.

Therefore, by similar triangles, if we drop a perpendicular from O to the line segment AC, the perpendicular will bisect AC. Therefore the length of AP (and PC, of course) is $R/2$.

We now have a right−angled triangle and enough information to calculate the relationship between r and R:

$$\cos x = R/(2r) \quad (1)$$

$$R = 2r\cos x \quad (2)$$

So the total area accessible to the goat is:

$(4r^2\cos^2 x)x + r^2[\pi - 2x - \sin(2x)]$.

We wish for this area to be half the area of the total field; therefore we have:

$$4r^2 x\cos^2 x + r^2[\pi - 2x - \sin(2x)] = \pi r^2/2$$

$$4x\cos^2 x + \pi - 2x - \sin(2x) = \pi/2$$

$$4x\cos^2 x + \pi/2 - 2x - \sin(2x) = 0$$

We can't easily solve the equation but we can use a graphical calculator or numerical method such as Newton−Raphson to find an approximate solution.

Using the Newton−Raphson method as described in the Coda, we find that

x is approximately 0.953, and therefore $\cos x = 0.579$.

Now, since $R = 2r\cos x$ and r, the radius of the field, is 100m, we have $R = 200\cos x$ and thus the required length of rope is approximately 116m.

Coda: Solving the equation using Newton−Raphson

The basic idea

In the goat puzzle, we were left with the following equation to solve:

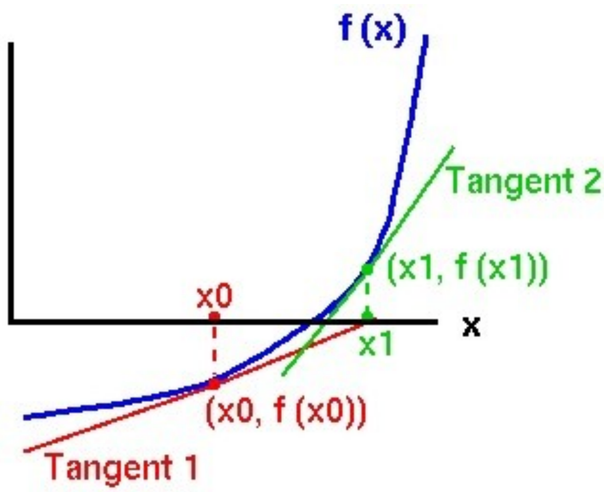
$4x\cos^2 x + \pi/2 - 2x - \sin(2x) = 0$

The Newton−Raphson method is an approximate method for finding roots of equations that are differentiable.

Let $f(x)$ be a differentiable function. Since $f(x)$ is differentiable, every point on the graph of $f(x)$ must have a gradient and a unique tangent line.

Now, the tangent at x_0 is an approximation to the graph of $f(x)$ near the point $(x_0, f(x_0))$.

Therefore the zero of the tangent line (the point where the tangent line crosses the x−axis) is an approximation (perhaps a very bad one, however!) of the zero of $f(x)$ (the point where $f(x)$ crosses the x−axis, i.e. the root of $f(x)$). Itâ€™s like weâ€™re pretending that $f(x)$ is really a straight line, like the tangent line, and therefore crosses the x−axis at the same place the tangent does.



In the Newton−Raphson method, we start with a "best guess" x_0 as to the zero of $f(x)$. We then calculate the first approximation, x_1, as the zero of the tangent line to $f(x)$ at x_0.

We then calculate the second approximation, x_2, as the zero of the tangent line crossing the x−axis at x_1, and so forth.

The diagram above shows the initial guess x_0, the first approximations x_1 and the relevant tangents. The second approximation x_2 is the coordinate where the second tangent crosses the x−axis. As you can see, the approximations are getting closer to the actual zero point of $f(x)$. If we continue iterating like this, we will get better and better estimates for the zero point of $f(x)$.

How do we do it?

We wish to solve $4x\cos^2 x + \pi/2 - 2x - \sin(2x) = 0$. Obviously, plotting $f(x) = 4x\cos^2 x + \pi/2 - 2x - \sin(2x)$ and drawing tangents is not going to be very much fun! However, we can perform Newton−Raphson numerically.

Our initial point is x_0. The gradient of $f(x)$ at x_0 is given by $f'(x_0)$, and the tangent line to $f(x)$ at x_0 is therefore given by:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

To find x_1, we must find the point where this tangent crosses the x−axis, i.e. to let:

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

and therefore

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)}$$

so that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, in the general case we obtain:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Now, our function is $f(x) = 4x\cos^2 x + \pi/2 - 2x - \sin(2x)$. Via standard differential calculus, the gradient $f'(x)$ of this function is

$$4\cos^2 x - 8x\cos x \sin x - 2 - 2\sin(2x)\cos(2x).$$

Therefore, to find the approximate root of $f(x)$ we can use the following:

$$x_{n+1} = x_n - \frac{4x_n\cos^2 x_n + \pi/2 - 2x_n - \sin(2x_n)}{4\cos^2 x_n - 8x_n\cos x_n \sin x_n - 2 - 2\sin(2x_n)\cos(2x_n)}$$

So, we know how to calculate x_{n+1} from x_n. But how do we find our starting value, x_0? Well, in this particular case we know that the magnitude of x must be between 0 and $\pi/2$ radians (go back to the second diagram and think about it if youâ€™re not sure why!). So a good initial guess might be (for example) $\pi/4$.

As it turns out, all sorts of values will do: hereâ€™s a table of the iterative steps of Newton−Raphson on our function $f(x)$ for a range of initial values of x_0. As you can see, they all converge quite rapidly to the same twelve−significant−digit approximation.

	$x_0 = \pi/4$	$x_0 = \pi/6$	$x_0 = \pi/3$	$x_0 = \pi/2$
x_0	0.785398163397	0.523598775598	1.047197551200	1.570796326790
x_1	0.967088277214	1.254847487960	0.956164730983	0.785398163397
x_2	0.953058379193	0.966611488070	0.952884951928	0.967088277214
x_3	0.952849994306	0.953049230238	0.952848237620	0.953058379193
x_4	0.952847886046	0.952849901115	0.952847868401	0.952849994306
x_5	0.952847864870	0.952847885110	0.952847864693	0.952847886046
x_6	0.952847864657	0.952847864860	0.952847864655	0.952847864870
x_7	0.952847864655	0.952847864657	0.952847864655	0.952847864657
x_8	0.952847864655	0.952847864655	0.952847864655	0.952847864655
x_9	0.952847864655	0.952847864655	0.952847864655	0.952847864655

K.E.M.

Plus is part of the family of activities in the Millennium Mathematics Project, which also includes the NRICH and MOTIVATE sites.