

Figure 1: Triangles drawn on a sphere have angles adding to more than 180 degrees. Image: Lars H. Rohwedder.

Over 2000 years ago the Greek mathematician Euclid set out a list of five postulates on which he thought geometry should be built. One of them, the fifth, was equivalent to a statement we are all familiar with: that the angles in a triangle add up to 180 degrees. However, this postulate did not seem as obvious as the other four on Euclid's list, so mathematicians attempted to deduce it from them: to show that a geometry obeying the first four postulates would necessarily obey the fifth. Their struggle continued for centuries, but in the end they failed. They found examples of geometries that do not obey the fifth postulate.

## Spherical geometry

In spherical geometry the Euclidean idea of a line becomes a *great circle*, that is, a circle of maximum radius spanning the fattest part of the Earth. It is no longer true that the sum of the angles of a triangle is always 180 degrees. Very small triangles will have angles summing to only a little more than 180 degrees (because, from the perspective of a very small triangle, the surface of a sphere is nearly flat). Bigger triangles will have angles summing to very much more than 180 degrees.

**When we first encounter geometry at school, we learn an axiom laid down by the Greek mathematician Euclid over 2,000 years ago: that the angles in a triangle add up to 180 degrees. But in non-Euclidean geometries this is not always true.**

One funny thing about the length of time it took to discover spherical geometry is that it is the geometry that holds on the surface of the Earth! But we never really notice, because we are so small compared to the size of the Earth that if we draw a triangle on the ground, and measure its angles, the amount by which the sum of the angles exceeds 180 degrees is so tiny that we can't detect it.

But is there a geometry in which the angles of a triangle sum to *less than* 180 degrees? The answer is yes:

## Hyperbolic geometry

Hyperbolic geometry isn't as easy to visualise as spherical geometry because it can't be modelled in three-dimensional Euclidean space without distortion. One way of visualising it is called the *Poincaré disc*.

Take a round disc and imagine an ant living within it. In Euclidean geometry the shortest path between two points inside that disc is along a straight line. In hyperbolic geometry distances are measured differently so the shortest path is no longer along a Euclidean straight line but along the arc of a circle that meets the boundary of the disc at right angles. A hyperbolic ant would experience the straight-line path as a detour – it prefers to move along the arc of such a circle.

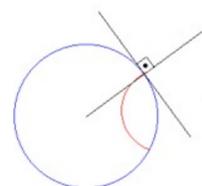


Figure 2: The red circle meets the boundary circle at right angles. For a hyperbolic ant living inside the disc bounded by the blue circle the path of shortest distance between two points is along the arc of such a circle.

A hyperbolic triangle, whose sides are arcs of these semicircles, has angles that add up to less than 180 degrees.

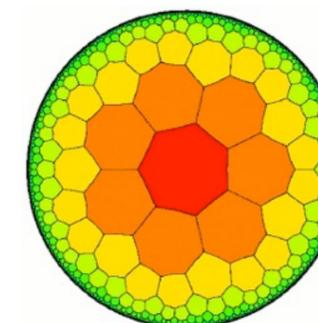


Figure 3: All the black and white shapes in this image are hyperbolic triangles with angles adding to less than 180 degrees.

One consequence of this new hyperbolic metric is that the boundary circle of the disc is infinitely far away from the point of view of the hyperbolic ant. This is because the metric distorts distances with respect to the ordinary Euclidean one. Paths that look the same length in the Euclidean metric are longer in the hyperbolic metric the closer they are to the boundary circle.

Figure 4 shows a tiling of the hyperbolic plane by regular heptagons. Because of the distorted metric the heptagons are all of the same size in the hyperbolic metric. And as we can see the ant would need to traverse infinitely many of them to get to the boundary circle – it is infinitely far away!

Figure 4: A tiling of the Poincaré disc by hyperbolic heptagons. (Image: David Wright)



Hyperbolic geometry may look like a fanciful mathematical construct but it has real-life uses. When Einstein developed his special theory of relativity in 1905 he found that the symmetries of hyperbolic geometry were exactly what he needed to formulate the theory. Today mathematicians believe that hyperbolic geometry may help to understand large networks like Facebook or the Internet.

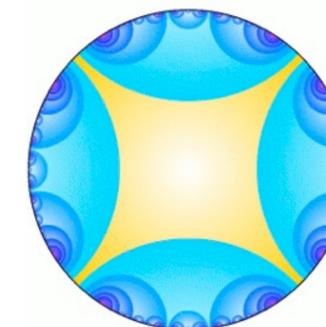


Figure 5: Another tiling of the hyperbolic plane, this time by eight-sided shapes that have four of their sides on the boundary circle. (Image: David Wright)



This poster is based on some of the content in the articles 'Strange Geometries' by Dr Helen Joyce and 'Non-Euclidean Geometry and Indra's Pearls' by Professor Caroline Series and Professor David Wright, published in Plus ([plus.maths.org](http://plus.maths.org)), a free online magazine opening a door to the fascinating world of mathematics.