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Features

Mysterious number 6174

by Yutaka Nishiyama

Anyone can uncover the mystery

The number 6174 is a really mysterious number. At first glance, it might not seem so obvious. But as we are about to see, anyone who can subtract can uncover the mystery that makes 6174 so special.

Kaprekar's operation

In 1949 the mathematician D. R. Kaprekar from Devlali, India, devised a process now known as *Kaprekar's operation*. First choose a four digit number where the digits are not all the same (that is not 1111, 2222,...). Then rearrange the digits to get the largest and smallest numbers these digits can make. Finally, subtract the smallest number from the largest to get a new number, and carry on repeating the operation for each new number.

It is a simple operation, but Kaprekar discovered it led to a surprising result. Let's try it out, starting with the number 2005, the digits of last year. The maximum number we can make with these digits is 5200, and the minimum is 0025 or 25 (if one or more of the digits is zero, embed these in the left hand side of the minimum number). The subtractions are:

$$
\begin{aligned}
5200 - 0025 &= 5175 \\
7551 - 1557 &= 5994 \\
9954 - 4599 &= 5355 \\
5553 - 3555 &= 1998 \\
9981 - 1899 &= 8082 \\
8820 - 0288 &= 8532 \\
8532 - 2358 &= 6174 \\
7641 - 1467 &= 6174
\end{aligned}
$$

When we reach 6174 the operation repeats itself, returning 6174 every time. We call the number 6174 a *kernel* of this operation. So 6174 is a kernel for Kaprekar's operation, but is this as special as 6174 gets? Well not only is 6174 the only kernel for the operation, it also has one more surprise up it's sleeve. Let's try again starting with a different number, say 1789.

$$
\begin{aligned}
9871 - 1789 &= 8082 \\
8820 - 0288 &= 8532 \\
8532 - 2358 &= 6174
\end{aligned}
$$

We reached 6174 again!

A very mysterious number...

When we started with 2005 the process reached 6174 in seven steps, and for 1789 in three steps. In fact, you reach 6174 for all four digit numbers that don't have all the digits the same. It's marvellous, isn't it? Kaprekar's operation is so simple but uncovers such an interesting result. And this will become even more intriguing when we think about the reason why all four digit numbers reach this mysterious number 6174.

Only 6174?

The digits of any four digit number can be arranged into a maximum number by putting the digits in descending order, and a minimum number by putting them in ascending order. So for four digits *a,b,c,d* where

$$9 \quad a \quad b \quad c \quad d \quad 0$$

and *a, b, c, d* are not all the same digit, the maximum number is *abcd* and the minimum is *dcba*.

We can calculate the result of Kaprekar's operation using the standard method of subtraction applied to each column of this problem:

$$\begin{array}{rcccc} & a & b & c & d \\ - & d & c & b & a \\ \hline & A & B & C & D \end{array}$$

which gives the relations

$$\begin{aligned} D &= 10 + d - a \text{ (as } a > d) \\ C &= 10 + c - 1 - b = 9 + c - b \text{ (as } b > c - 1) \\ B &= b - 1 - c \text{ (as } b > c) \\ A &= a - d \end{aligned}$$

for those numbers where $a>b>c>d$.

A number will be repeated under Kaprekar's operation if the resulting number *ABCD* can be written using the initial four digits *a,b,c* and *d*. So we can find the kernels of Kaprekar's operation by considering all the possible combinations of {*a, b, c, d*} and checking if they satisfy the relations above. Each of the 4! = 24 combinations gives a system of four simultaneous equations with four unknowns, so we should be able to solve this system for *a, b, c* and *d*.

It turns out that only one of these combinations has integer solutions that satisfy *9 a b c d 0*. That combination is *ABCD = bdac*, and the solution to the simultaneous equations is *a*=7, *b*=6, *c*=4 and *d*=1. That is *ABCD* = 6174. There are no valid solutions to the simultaneous equations resulting from some of the digits in *{a,b,c,d}* being equal. Therefore the number 6174 is the only number unchanged by Kaprekar's operation our mysterious number is unique.

For three digit numbers the same phenomenon occurs. For example applying Kaprekar's operation to the three digit number 753 gives the following:

$$753 - 357 = 396$$
$$963 - 369 = 594$$
$$954 - 459 = 495$$
$$954 - 459 = 495$$

The number 495 is the unique kernel for the operation on three digit numbers, and all three digit numbers reach 495 using the operation. Why don't you check it yourself?

How fast to 6174?

It was about 1975 when I first heard about the number 6174 from a friend, and I was very impressed at the time. I thought that it would be easy to prove why this phenomenon occurred but I could not actually find the reason why. I used a computer to check whether all four digit numbers reached the kernel 6174 in a limited number of steps. The program, which was about 50 statements in Visual Basic, checked all of 8991 four digit numbers from 1000 to 9999 where the digits were not all the same.

The table below shows the results: every four digit number where the digits aren't all equal reaches 6174 under Kaprekar's process, and in at most seven steps. If you do not reach 6174 after using Kaprekar's operation seven times, then you have made a mistake in your calculations and should try it again!

Iteration	**Frequency**
0	1
1	356
2	519
3	2124
4	1124
5	1379
6	1508
7	1980

Which way to 6174?

My computer program checked all 8991 numbers, but in his article Malcolm Lines explains that it is enough to check only 30 of all the possible four digit numbers when investigating Kaprekar's operation.

As before let's suppose that the four digit number is *abcd*, where

9 a b c d 0.

Let us calculate the first subtraction in the process. The maximum number is *1000a+100b+10c+d* and the minimum number is *1000d+100c+10b+a*. So the subtraction is:

$$
\begin{aligned}
&1000a + 100b + 10c + d - (1000d + 100c + 10b + a)\\
&= 1000(a-d) + 100(b-c) + 10(c-b) + (d-a)\\
&= 999(a-d) + 90(b-c)
\end{aligned}
$$

The possible value of $(a-d)$ is from 1 to 9, and $(b-c)$ is from 0 to 9. By running through all the possibilities, we can see all the possible results from the first subtraction in the process. These are shown in Table 1.

		999X(a−d)								
		1	2	3	4	5	6	7	8	9
	0	999	1998	2997	3996	4995	5994	6993	7992	8991
	1	1089	2088	3087	4086	5085	6084	7083	8082	9081
	2	1179	2178	3177	4176	5175	6174	7173	8172	9171
	3	1269	2268	3267	4266	5265	6264	7263	8262	9261
90X	4	1359	2358	3357	4356	5355	6354	7353	8352	9351
(b−c)	5	1449	2448	3447	4446	5445	6444	7443	8442	9441
	6	1539	2538	3537	4536	5535	6534	7533	8532	9531
	7	1629	2628	3627	4626	5625	6624	7623	8622	9621
	8	1719	2718	3717	4716	5715	6714	7713	8712	9711
	9	1809	2808	3807	4806	5805	6804	7803	8802	9801

Table 1: Numbers after the first subtraction in Kaprekar's process

We are only interested in numbers where the digits are not all equal and

$$a \quad b \quad c \quad d,$$

therefore we only need to consider those where $(a-d) \quad (b-c)$. So we can ignore the grey region in Table 1 which contains those numbers where

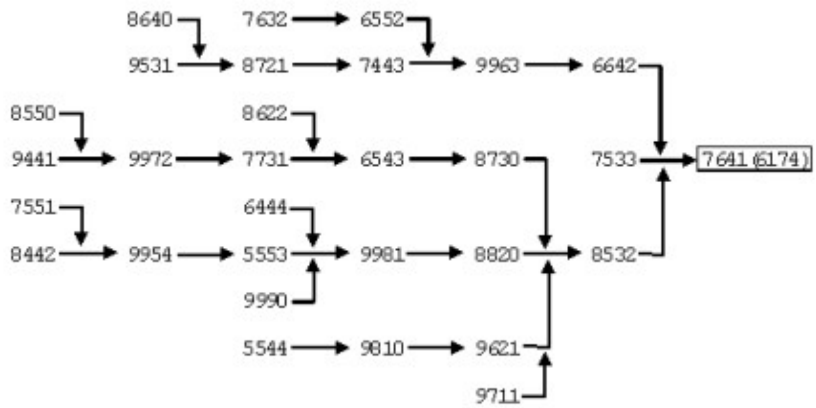
$$(a-d) < (b-c).$$

Now we arrange the digits of the numbers in the table in descending order, to get the maximum number ready for the second subtraction:

		999X(a−d)								
		1	2	3	4	5	6	7	8	9
	0	9990	9981	9972	9963	9954	9954	9963	9972	9981
	1	9810	8820	8730	8640	8550	8640	8730	8820	9810
	2		8721	7731	7641	7551	7641	7731	8721	9711
	3			7632	6642	6552	6642	7632	8622	9621
90X	4				6543	5553	6543	7533	8532	9531
(b−c)	5					5544	6444	7443	8442	9441
	6						6543	7533	8532	9531
	7							7632	8622	9621
	8								8712	9711
	9									9801

Table 2: Maximum numbers, ready for the second subtraction

We can ignore the duplicates in Table 2 (the grey regions), and are left with just 30 numbers to follow through the rest of the process. The following figure shows the routes which these numbers take to reach 6174.



How these 30 numbers reach 6174

From this figure you can see how all the four digit numbers reach 6174 and reach it in at most seven steps. Even so I still think it is very mysterious. I guess Kaprekar, who discovered this number, was extremely clever or had a lot of time to think about it!

Two digits, five digits, six and beyond...

We have seen that four and three digit numbers reach a unique kernel, but how about other numbers? It turns out that the answers for those is not quite as impressive. Let try it out for a two digit number, say 28:

$$\begin{aligned}82 - 28 &= 45\\ 54 - 45 &= 9\\ 90 - 09 &= 81\\ 81 - 18 &= 63\\ 63 - 36 &= 27\\ 72 - 27 &= 45\\ 54 - 45 &= 9\end{aligned}$$

It doesn't take long to check that all two digit numbers will reach the loop 9816327459. Unlike for three and four digit numbers, there is no unique kernel for two digit numbers.

But what about five digits? Is there a kernel for five digit numbers like 6174 and 495? To answer this we would need to use a similar process as before: check the 120 combinations of $\{a,b,c,d,e\}$ for $ABCDE$ such that

$$9 \quad a \quad b \quad c \quad d \quad e \quad 0$$

and

$$abcde - edcba = ABCDE.$$

Thankfully the calculations have already been done by a computer, and it is known that there is no kernel for Kaprekar's operation on five digit numbers. But all five digit numbers do reach one of the following three loops:

71973839527494362964719 73
75933639546197482962759 33

599945395559994

As Malcolm Lines points out in his article, it will take a lot of time to check what happens for six or more digits, and this work becomes extremely dull! To save you from this fate, the following table shows the kernels for two digit to ten digit numbers (for more see Mathews Archive of Recreational Mathematics). It appears that Kaprekar's operation takes every number to a unique kernel only for three and four digit numbers.

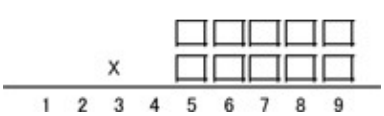
Digits	Kernel
2	None
3	495
4	6174
5	None
6	549945, 631764
7	None
8	63317664, 97508421
9	554999445, 864197532
10	6333176664, 9753086421, 9975084201

Beautiful, but is it special?

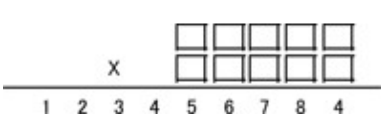
We have seen that all three digit numbers reach 495, and all four digit numbers reach 6174 under Kaprekar's operation. But I have not explained why all such numbers reach a unique kernel. Is this phenomenon incidental, or is there some deeper mathematical reason why this happens? Beautiful and mysterious as the result is, it might just be incidental.

Let's stop and consider a beautiful puzzle by Yukio Yamamoto in Japan.

If you multiply two five digit numbers you can get the answer 123456789. Can you guess the two five digit numbers?



This is a very beautiful puzzle and you might think that a big mathematical theory should be hidden behind it. But in fact it's beauty is only incidental, there are other very similar, but not so beautiful, examples. Such as:



(We can give you a hint to help you solve these puzzles, and here are the answers.)

If I showed you Yamamoto's puzzle you would be inspired to solve it because it is so beautiful, but if I showed you the second puzzle you might not be interested at all. I think Kaprekar's problem is like Yamamoto's number guessing puzzle. We are drawn to both because they are so beautiful. And because they are so beautiful we feel there must be something more to them when in fact their beauty may just be incidental. Such misunderstandings have led to developments in mathematics and science in the past.

Is it enough to know all four digit numbers reach 6174 by Kaprekar's operation, but not know the reason why? So far, nobody has been able to say that all numbers reaching a unique kernel for three and four digit numbers is an incidental phenomenon. This property seems so surprising it leads us to expect that a big theorem in number theory hides behind it. If we can answer this question we could find this is just a beautiful misunderstanding, but we hope not.

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About the author

Yutaka Nishiyama is a professor at Osaka University of Economics, Japan. After studying mathematics at the University of Kyoto he went on to work for IBM Japan for 14 years. He is interested in the mathematics that occurs in daily life, and has written seven books about the subject. The most recent one, called "The mystery of five in nature", investigates, amongst other things, why many flowers have five petals. Professor Nishiyama is currently visiting the University of Cambridge.

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