Euler's polyhedron formula

by Abigail Kirk

No simple polyhedron has seven edges.

Proof: We show first that for any polyhedron we have $2E \leq 3F$ and $2E \leq 3V$. The faces of the polyhedron are polygons, each bounded by a number of sides. Along each edge exactly two faces come together, so an edge corresponds to exactly two sides: the total number of sides is $2E$. We also notice that any face has at least 3 sides, so the total number of sides is at least $3F$ times the number of faces. Thus we get:

\[
\text{The total number of sides} = 2E
\]

and

\[
\text{The total number of sides} \geq 3F.
\]

Putting this together we get:

\[
2E \leq 3F,
\]

proving our first inequality.

To prove the second inequality we count the total number of ends of edges. Each edge has two ends, so the total number of ends equals $2E$. At each vertex at least three edges come together, so the total number of ends of edges is at least 3 times the number of vertices. Putting this together we get:
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Now, if a polyhedron has 7 edges, then $3F \leq 14$ and $3V \leq 14$. This means that both $F$ and $V$ cannot be bigger than 4. A little thought will convince you that every polyhedron has strictly more than three faces, so we must have $F=4$. Similarly we get that $V=4$. This gives

$$V - E + F = 4 - 7 + 4 = 1 \neq 2.$$  

This tells us that our hypothetical seven−edged polyhedron cannot exist, for if it did, then Euler's formula would hold and the above sum would have to be equal to 2  QED!

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