Probability Measures in Financial Mathematics

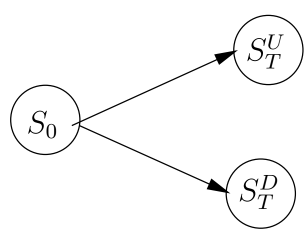
Arbitrage
Arbitrage is the ability to make a riskless profit. For example, say you see the following exchange rates:

GBP 1 buys	JPY 130.47,
GBP 1 buys	USD 1.466,
USD 1 buys	JPY 89.1.

You can use 1 GBP to buy 1.466 USD and use these USD to buy $(1.466 \times 89.1) = 130.62$ JPY. Now use these JPY to buy GBP, you end up with $\frac{130.62}{130.47} = 1.0012 > 1$ and you have made a riskless profit of 0.12 pence.

In derivative pricing the basic principle is to price assets so that no arbitrage profits can be made. This is because in financial markets, the price you quote is both the buying and selling price, and so both over and underpricing will enable others to profit from your mis-pricing.

A simple market
Consider a market that consists of a single asset and a single period. At time $t = 0$ the asset's price is S_0 and at time T the asset's price will be either S_T^U or S_T^D (the probability of the asset price going to either S_T^U or S_T^D is not known and may not be constant).



Say $S_T^D = S_0 < S_T^U$ then buying the bet will definitely not lose money, and may possibly make money - which is an arbitrage. If $S_T^D < S_0 = S_T^U$ the arbitrage exists by *selling* the asset. So, for there to be no arbitrage opportunities in this simple market, we require that

$$S_T^D < S_0 < S_T^U \qquad (\text{ or } S_T^D > S_0 > S_T^U).$$

Say a business is exposed to the risk of the asset's price moving and wants to use a financial product to reduce the risk. They ask you to supply this product, a derivative because its value is will be derived from the original asset's value, who's value at time $t = T$ will be either f_T^U or f_T^D. What is the no-arbitrage price, f_0, of this product?

We do not know the probability of the derivative value being f_T^U or f_T^D, and so it is not reasonable to price on "natural" expectations. The way we do price is by constructing a portfolio based on 1 derivative and a holding, δ (positive or negative), in the original asset. We determine δ by making the portfolio riskless, i.e.

$$\begin{aligned} f_T^U + \delta S_T^U &= f_T^D + \delta S_T^D \\ \Rightarrow \delta &= \frac{f_T^D - f_T^U}{S_T^U - S_T^D} \end{aligned}$$

Say we have sold the derivative at time $t = 0$ we can consider this as a gain of f_0, but having sold the derivative, we need to pay out on it at time $t = T$, a loss of either f_T^U or f_T^D. Applying this intuition with the holding in the underlying asset and the knowledge that, since our portfolio is riskless, we cannot make any money without creating an arbitrage, we have

$$(f_0 + \delta S_0) - (f_T^U + \delta S_T^U) = 0 = (f_0 + \delta S_0) - (f_T^D + \delta S_T^D)$$

so, solving for f_0,

$$\begin{aligned} f_0 &= \delta S_T^U - \delta S_0 + f_T^U \\ &= \frac{S_T^U - S_0}{S_T^U - S_T^D} f_T^D + \frac{S_0 - S_T^D}{S_T^U - S_T^D} f_T^U . \end{aligned}$$

f_0 is the "fair price", or the price that does not allow anyone to make a riskless, or unfair, profit.

Risk neutral probabilities

Given the no-arbitrage condition, $S_T^D < S_0 < S_T^U$, we can regard

$$\frac{S_T^U - S_0}{S_T^U - S_T^D} \qquad \text{and} \qquad \frac{S_0 - S_T^D}{S_T^U - S_T^D}$$

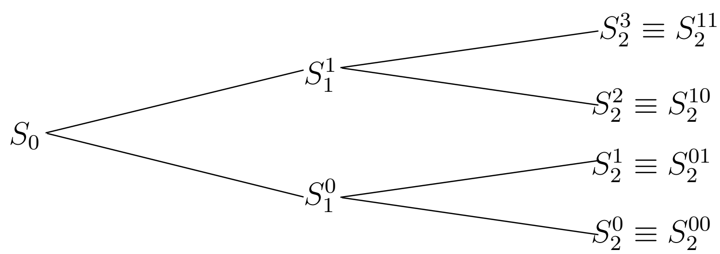
as special "probabilities", known as *risk neutral probabilities*, of a down and an up move, respectively, and we price assets by taking expectations using this measure, $f_0 = \mathbb{E}_{\mathbb{Q}}\big[f_T\big]$.

Incomplete markets

This analysis depends on the fact that there are only two possible terminal states of the world. If at time $t = T$ there were three or more states of the world, we could not establish a unique δ that makes our portfolio riskless. This situation is known as *market incompleteness*.

In a complete, arbitrage free market there is a unique risk probability measure. In incomplete, arbitrage free markets there is an infinite set of risk neutral measures.

Incompleteness can be resolved by generating a "tree" describing the asset's price evolution:



Each branch of the tree is complete, and so the tree as a whole is complete. Critical to this approach is that we have to specify the way the asset prices go from S_0 to $S_1^{\cdot}$ and to $S_2^{\cdot}$; the asset price model completes an incomplete market. Change this model and we change the risk neutral probabilities, change these probabilities and we change the derivative's price.

In reality, we cannot identify a model to describe the asset's price evolution accurately, and in reality all asset pricing is in incomplete markets. So, in practice, there is an infinite set of risk neutral probability measures and the "best one" must be chosen for pricing. However, none of these risk neutral probability measures will completely remove risk in the way it is removed in complete markets. This point has not fully permeated from financial maths into finance, where there is often an assumption that assets can be priced exactly and risk removed completely.